Modeling, Mathematical and Numerical Analysis for some Compressible and Incompressible Equations in Thin Layer.

M. Ersoy

15 october 2010

Outline of the talk

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1. **Introduction**
   - Atmosphere dynamic
   - Sedimentation
   - Unsteady mixed flows in closed water pipes
2. **Mathematical results on CPEs**
   - An intermediate model
   - Toward an existence result for the 2D-CPEs
   - Toward a stability result for the 3D-CPEs
   - Perspectives
3. **An upwinded kinetic scheme for the PFS equations**
   - Finite Volume method
   - Kinetic Formulation and numerical scheme
   - Numerical results
   - Perspectives
4. **Formal derivation of a SVEs like model**
   - A nice coupling : Vlasov and Anisotropic Navier-Stokes equations
   - Hydrodynamic limit, toward a "mixed model"
   - A Viscous Saint-Venant-Exner like model
   - Perspective

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Hydrostatic approximation and averaged equations

Navier Stokes equations (NSEs) or Euler equations (EEs) on
$\Omega = \{(x,y) \in \mathbb{R}^3; H \ll L\}$ "thin layer domain"

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Hydrostatic approximation (asymptotic analysis with $\varepsilon = H/L = W/V \ll 1$ and rescaling $\tilde{x} = x/L$, $\tilde{y} = y/H$, $\tilde{u} = u/U$ $\tilde{w} = w/W$)$\longrightarrow$ Primitive equations (PEs)

J. Pedlowski
Geophysical Fluid Dynamics.
2nd Edition, Springer-Verlag, New-York, 1987.

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$\downarrow$ [GP]

Averaged PEs with respect to depth or altitude $y \longrightarrow$ Saint-Venant Equations (SVEs)

J. Pedlowski
Geophysical Fluid Dynamics.
2nd Edition, Springer-Verlag, New-York, 1987.

J.-F Gerbeau and B. Perthame
Derivation of viscous Saint-Venant system for laminar shallow water ; numerical validation.
Discrete Contin. Dyn. Syst. Ser. B, 1(1), 2001.

Outline

Atmosphere dynamic

- Dynamic :
  - Compressible fluid
  - Small vertical extension with respect to horizontal
  - Principally horizontal movements
  - Density highly stratified

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  Hydrostatic approximation $\longrightarrow$ compressible primitive equations (CPEs)

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\begin{aligned}
\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) + \partial_y(\rho v) &= 0 \\
\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \partial_y(\rho \mathbf{u} v) + \nabla_x p &= \mathrm{div}_x(\sigma_x) + f \\
\partial_t(\rho v) + \mathrm{div}_x(\rho \mathbf{u} v) + \partial_y(\rho v^2) + \partial_y p &= -\rho g + \mathrm{div}_y(\sigma_y) \\
p &= c^2 \rho
\end{aligned}
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Atmosphere dynamic

- Dynamic :
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  - Density highly stratified $p = \xi(t,x)e^{-g/c^2 y}$
- Modeling : Compressible Navier-Stokes equations

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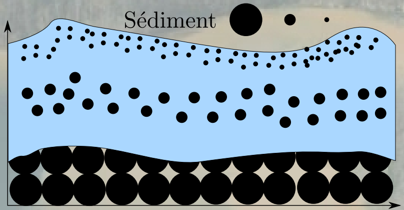
Outline

Sedimentation

- Sediment : produced by erosion process



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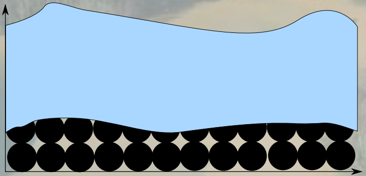

- Dynamic :
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  - Small vertical extension with respect to horizontal
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- Modeling : Saint-Venant-Exner equations
  - hydrodynamic part $\longrightarrow$ Saint-Venant equations (averaged IPEs)
  - morphodynamic part $\longrightarrow$ Exner equations

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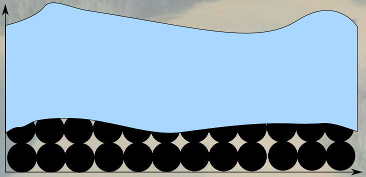

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$$\begin{cases} \partial_t h + \mathrm{div}(q) = 0, \\ \partial_t q + \mathrm{div}\left(\dfrac{q \otimes q}{h}\right) + \nabla\left(g\dfrac{h^2}{2}\right) = -gh\nabla b \end{cases}$$

  - morphodynamic part $\longrightarrow$ Exner equations

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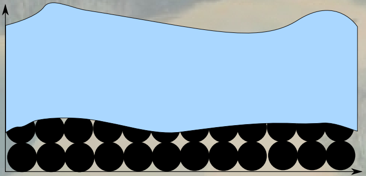


- Dynamic :
  - Incompressible fluid
  - Small vertical extension with respect to horizontal
  - Principally horizontal movements
  - variable bottom, example : bed river
- Modeling : Saint-Venant-Exner equations
  - hydrodynamic part $\longrightarrow$ Saint-Venant equations (averaged IPEs)

$$\left\{ \begin{array}{l} \partial_t h + \mathrm{div}(q) = 0, \\ \partial_t q + \mathrm{div}\left(\dfrac{q \otimes q}{h}\right) + \nabla\left(g\dfrac{h^2}{2}\right) = -gh\nabla b \end{array} \right.$$

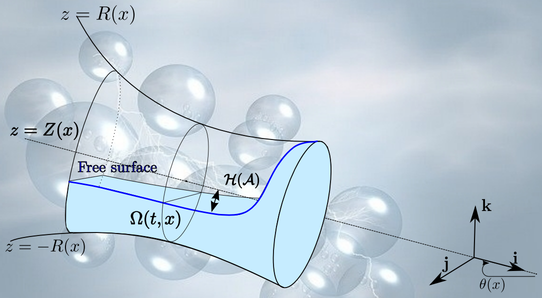
  - morphodynamic part $\longrightarrow$ Exner equations

$$\partial_t b + \xi \mathrm{div}(q_b(h, q)) = 0$$

Outline

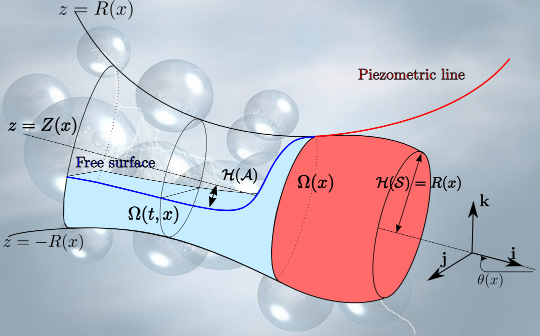
Unsteady mixed flows in closed water pipes

- mixed : Free surface and pressurized flows
  - Free Surface area (FS)
    Section non filled and incompressible flow. . .



Unsteady mixed flows in closed water pipes

- mixed : Free surface and pressurized flows
  - Free Surface area (FS)
    Section non filled and incompressible flow. . .
  - Pressurized area (P)
    Section completely filled and compressible flow. . .



- Dynamic :
  - Incompressible or compressible fluid following the area
  - Small vertical extension with respect to horizontal
  - Principally horizontal movements : unidirectional

Unsteady mixed flows in closed water pipes

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- Modeling : A nice coupling of Saint-Venant like equations
  - free surface part $\longrightarrow$ usual Saint-Venant equations
  - pressurized part $\longrightarrow$ Saint-Venant like equations

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  - free surface part $\longrightarrow$ usual Saint-Venant equations

$$\left\{\begin{array}{rcl} \partial_t A_{fs} + \partial_x Q_{fs} & = & 0, \\ \partial_t Q_{fs} + \partial_x \left(\dfrac{Q_{fs}^2}{A_{fs}} + p_{fs}(x, A_{fs}) \right) & = & -g A_{fs} \dfrac{d\ Z}{dx} + Pr_{fs}(x, A_{fs}) - G(x, A \\ & & -K(x, A_{fs}) \dfrac{Q_{fs}|Q_{fs}|}{A_{fs}} \end{array}\right.$$

  - pressurized part $\longrightarrow$ Saint-Venant like equations

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$$\left\{\begin{array}{rcl} \partial_t A_p + \partial_x Q_p & = & 0, \\ \partial_t Q_p + \partial_x \left(\dfrac{Q_p^2}{A_p} + p_p(x, A_p) \right) & = & -gA_p \dfrac{d\,Z}{dx} + Pr_p(x, A_p) - G(x, A_p) \\ & & -K(x, A_p) \dfrac{Q_p|Q_p|}{A_p} \end{array}\right.$$

Unsteady mixed flows in closed water pipes

- Dynamic :
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- Modeling : A nice coupling : The PFS model
  - from the coupling :

$$\begin{array}{lcll} A & = & \left\{\begin{array}{lll} A_{fs} & \text{if} & \text{FS} \\ A_p & \text{if} & \text{P} \end{array}\right. & : \quad \text{the mixed variable} \\ Q & = & Au & : \quad \text{the discharge} \end{array}$$

$$\downarrow$$

$$\left\{\begin{array}{ll} \partial_t(A) + \partial_x(Q) & = 0 \\ \partial_t(Q) + \partial_x\left(\dfrac{Q^2}{A} + p(x,A,E)\right) & = -g\,A\,\dfrac{d}{dx}Z(x) \\ & \quad + Pr(x,A,E) \\ & \quad - G(x,A,E) \\ & \quad - g\,K(x,\mathbf{S})\,\dfrac{Q|Q|}{A} \end{array}\right.$$

where E is a state indicator and appropriate p and Pr

Outline

Energy estimates ?

CPEs :

$$\left\{\begin{array}{l}\partial_t\rho + \mathrm{div}_x\left(\rho\,\mathbf{u}\right) + \partial_y\left(\rho v\right) = 0,\\ \partial_t\left(\rho\,\mathbf{u}\right) + \mathrm{div}_x\left(\rho\,\mathbf{u}\otimes\mathbf{u}\right) + \partial_y\left(\rho\,v\mathbf{u}\right) + \nabla_x p(\rho) = 2\mathrm{div}_x\left(\nu_1 D_x(\mathbf{u})\right) + \partial_y\left(\nu_2\partial_y\mathbf{u}\right),\\ \partial_y p(\rho) = -g\rho\\ p(\rho) = c^2\rho\end{array}\right.$$

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Problem : How to obtain energy estimates since : the sign of

$$\int_\Omega \rho g v\,dxdy$$

$$\frac{d}{dt}\int_\Omega \rho|u|^2+\rho\ln\rho-\rho+1\,dxdy+\int_\Omega 2\nu_1|D_x(u)|^2+\nu_2|\partial_y^2 u|\,dxdy+\int_\Omega \rho g v\,dxdy=0$$

is unknown !

Energy estimates ?

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$$\int_\Omega \rho g v \, dxdy$$

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is unknown !

Consequently standard techniques fails

Outline

The key point : the hydrostatic equation

Using the hydrostatic equation, we obviously have :

$$\rho(t, x, y) = \xi(t, x)e^{-g/c^2 y}$$

for some function $\xi(t, x)$: ρ is stratified

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Problem : find equations satisfied by ξ

An intermediate model :

- replace ρ by $\xi e^{-g/c^2 y}$ in CPEs

$$\left\{\begin{array}{l} \partial_t(\xi e^{-g/c^2 y}) + \mathrm{div}_x\left(\xi e^{-g/c^2 y}\,\mathbf{u}\right) + \partial_y\left(\xi e^{-g/c^2 y} v\right) = 0, \\ \partial_t\left(\xi e^{-g/c^2 y}\,\mathbf{u}\right) + \mathrm{div}_x\left(\xi e^{-g/c^2 y}\,\mathbf{u}\otimes\mathbf{u}\right) + \partial_y\left(\xi e^{-g/c^2 y}\, v\mathbf{u}\right) \\ +\nabla_x c^2 \nabla_x(\xi e^{-g/c^2 y}) = 2\mathrm{div}_x\left(\nu_1 D_x(\mathbf{u})\right) + \partial_y\left(\nu_2 \partial_y \mathbf{u}\right), \\ \rho = \xi e^{-g/c^2 y} \end{array}\right.$$

- multiply CPEs by $e^{+g/c^2 y}$

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An intermediate model :

- replace ρ by $\xi e^{-g/c^2 y}$ in CPEs
- multiply CPEs by $e^{+g/c^2 y}$

$$\begin{cases} \partial_t(\xi) + \mathrm{div}_x(\xi\,\mathbf{u}) + e^{g/c^2 y}\partial_y\left(\xi e^{-g/c^2 y} v\right) = 0, \\ \partial_t(\xi\,\mathbf{u}) + \mathrm{div}_x(\xi\,\mathbf{u}\otimes\mathbf{u}) + e^{g/c^2 y}\partial_y\left(\xi e^{-g/c^2 y}\, v\mathbf{u}\right) + c^2\nabla_x\xi = \\ 2e^{g/c^2 y}\mathrm{div}_x(\nu_1 D_x(\mathbf{u})) + e^{g/c^2 y}\partial_y(\nu_2\partial_y\mathbf{u}), \\ \rho = \xi e^{-g/c^2 y} \end{cases}$$

- set $z = 1 - e^{-g/c^2 y}$ such that $e^{g/c^2 y}\partial_y = \partial_z$ and $w = e^{-g/c^2 y}v$ under suitable choice of viscosities.

The key point : the hydrostatic equation

Using the hydrostatic equation, we obviously have :

$$\rho(t,x,y) = \xi(t,x)e^{-g/c^2 y}$$

for some function $\xi(t,x)$: ρ is stratified
Problem : find equations satisfied by ξ
An intermediate model :

$$\left\{\begin{array}{l} \partial_t \xi + \mathrm{div}_x(\xi \mathbf{u}) + \xi \partial_z w = 0, \\ \partial_t(\xi \mathbf{u}) + \mathrm{div}_x\left(\xi\, \mathbf{u} \otimes \mathbf{u}\right) + \partial_z\left(\xi\, w \mathbf{u}\right) + c^2 \nabla_x(\xi) = 2\mathrm{div}_x\left(\nu_1 D_x(\mathbf{u})\right) + \partial_z\left(\nu_2 \partial_z \mathbf{u}\right), \\ \partial_z \xi = 0 \end{array}\right.$$

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$$\frac{d}{dt}\int_\Omega \xi|u|^2 + \xi \ln \xi - \xi + 1 \, dxdz + \int_\Omega 2\nu_1 |D_x(u)|^2 + \nu_2 |\partial_z^2 u| \, dxdz = 0$$

Outline

The 2D-CPEs

We set

$$\begin{cases} \nu_1(t,x,y) = \nu_0 e^{-g/c^2 y} \text{ for some given positive constant } \nu_0, \\ \nu_2(t,x,y) = \nu_1 e^{g/c^2 y} \text{ for some given positive constant } \nu_1. \end{cases}$$

the boundary conditions (BC)

$$\begin{cases} u_{|x=0} = u_{|x=l} = 0, \\ v_{|y=0} = v_{|y=h} = 0, \\ \partial_y u_{|y=0} = \partial_y u_{|y=h} = 0 \end{cases}$$

and the initial conditions (IC) :

$$\begin{cases} u_{|t=0} = u_0(x,y), \\ \rho_{|t=0} = \xi_0(x) e^{-g/c^2 y} \end{cases}$$

where ξ_0 :

$$0 < m \leqslant \xi_0 \leqslant M < \infty.$$

THEOREM ([EN2010])

Suppose that initial data (ξ_0, u_0) *have the properties :*

$$(\xi_0, u_0) \in W^{1,2}(\Omega), \quad u_{0|x=0} = u_{0|x=l} = 0.$$

Then $\rho(t,x,y)$ *is a bounded strictly positive function and the 2D-CPEs with BC has a weak solution in the following sense : a weak solution of 2D-CPEs with BC is a collection* (ρ, u, v) *of functions such that* $\rho \geqslant 0$ *and*

$$\rho \in L^\infty(0,T; W^{1,2}(\Omega)), \quad \partial_t \rho \in L^2(0,T; L^2(\Omega)),$$

$$u \in L^2(0,T; W^{2,2}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)), \quad v \in L^2(0,T; L^2(\Omega))$$

which satisfies the 2D-CPEs in the distribution sense ; in particular, the integral identity holds for all $\phi_{|t=T} = 0$ *with compact support :*

$$\begin{aligned} &\int_0^T \int_\Omega \rho u \partial_t \phi + \rho u^2 \partial_x \phi + \rho u v \partial_z \phi + \rho \partial_x \phi + \rho v \phi \, dxdydt \\ &= -\int_0^T \int_\Omega \nu_1 \partial_x u \partial_x \phi + \nu_2 \partial_y u \partial_y \phi \, dxdydt + \int_\Omega u_0 \rho_0 \phi_{|t=0} \, dxdy \end{aligned}$$

M. Ersoy and T. Ngom
Existence of a global weak solution to one model of Compressible Primitive Equations.
submitted to Applied Mathematics Letters, 2010.

THE PROOF

The intermediate model (IM) is exactly the model studied by Gatapov *et al* [GK05], derived from Equations 2D-CPEs by neglecting some terms, for which they provide the following global existence result :

Theorem (B. Gatapov and A.V. Kazhikhov 2005)

Suppose that initial data (ξ_0, u_0) *have the properties :*

$$(\xi_0, u_0) \in W^{1,2}(\Omega), \quad u_{0|x=0} = u_{0|x=1} = 0.$$

Then $\xi(t,x)$ *is a bounded strictly positive function and the IM has a weak solution in the following sense : a weak solution of the IM satisfying the BC is a collection* (ξ, u, w) *of functions such that* $\xi \geqslant 0$ *and*

$$\xi \in L^\infty(0,T;W^{1,2}(0,1)), \quad \partial_t\xi \in L^2(0,T;L^2(0,1)),$$

$$u \in L^2(0,T;W^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)), \quad w \in L^2(0,T;L^2(\Omega))$$

which satisfy the IM in the distribution sense.

B. V. Gatapov and A. V. Kazhikhov
Existence of a global solution to one model problem of atmosphere dynamics.
Sibirsk. Mat. Zh., pages 1011 :1020–722, 2005.

THE PROOF

By the simple change of variables $z = 1 - e^{-y}$ in the integrals, we get :

- $\| \rho \|_{L^2(\Omega)} = \alpha \| \xi \|_{L^2(\Omega)}$,
- $\| \nabla_x \rho \|_{L^2(\Omega)} = \alpha \| \nabla_x \xi \|_{L^2(\Omega)}$,
- $\| \partial_y \rho \|_{L^2(\Omega)} = \alpha \| \xi \|_{L^2(\Omega)}$

where $\alpha = \displaystyle\int_0^{1-e^{-1}} (1-z)\, dz < +\infty$. We deduce then,

$$\| \rho \|_{W^{1,2}(\Omega)} = \alpha \| \xi \|_{W^{1,2}(\Omega)}$$

which provides

$$\rho \in L^\infty(0,T;W^{1,2}(\Omega)) \text{ and } \partial_t \rho \in L^2(0,T;L^2(\Omega)).$$

$v \in L^2(0,T;L^2(\Omega))$ since the inequality holds :

$$\begin{aligned} \| v \|_{L^2(\Omega)} &= \int_0^1 \int_0^1 |v(t,x,y)|^2\, dy\, dx \\ &= \int_0^1 \int_0^{1-e^{-1}} \left(\frac{1}{1-z} \right)^3 |w(t,x,z)|^2\, dz\, dx \\ &< e^3 \| w \|_{L^2(\Omega)} . \end{aligned}$$

Finally, all estimates on u remain true. $\square$

OUTLINE

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   - Atmosphere dynamic
   - Sedimentation
   - Unsteady mixed flows in closed water pipes
2. MATHEMATICAL RESULTS ON CPES
   - An intermediate model
   - Toward an existence result for the 2D-CPEs
   - **Toward a stability result for the 3D-CPEs**
   - Perspectives
3. AN UPWINDED KINETIC SCHEME FOR THE PFS EQUATIONS
   - Finite Volume method
   - Kinetic Formulation and numerical scheme
   - Numerical results
   - Perspectives
4. FORMAL DERIVATION OF A SVES LIKE MODEL
   - A nice coupling : Vlasov and Anisotropic Navier-Stokes equations
   - Hydrodynamic limit, toward a "mixed model"
   - A Viscous Saint-Venant-Exner like model
   - Perspective

The 3D-CPEs

We set

$$\nu_1(t,x,y) = \bar{\nu}_1 \rho(t,x,y) \text{ and } \nu_2 = \bar{\nu}_2 \rho(t,x,y) e^{2y}.$$

for some positive constant $\bar{\nu}_1$ and $\bar{\nu}_2$.
We consider the IC and BC' where we prescribe periodic conditions on the spatiale domain with respect to x.
We define the set of function $\rho \in \mathcal{PE}(\mathbf{u}, v; y, \rho_0)$ such that

$$\begin{array}{ll}
\rho \in L^\infty(0,T;L^3(\Omega)), & \sqrt{\rho} \in L^\infty(0,T;H^1(\Omega)), \\
\sqrt{\rho}\mathbf{u} \in L^2(0,T;(L^2(\Omega))^2), & \sqrt{\rho} v \in L^\infty(0,T;L^2(\Omega)), \\
\sqrt{\rho} D_x(\mathbf{u}) \in L^2(0,T;(L^2(\Omega))^{2\times 2}), & \sqrt{\rho}\partial_y v \in L^2(0,T;L^2(\Omega)), \\
\nabla\sqrt{\rho} \in L^2(0,T;(L^2(\Omega))^3) &
\end{array}$$

with $\rho \geqslant 0$ and where $(\rho, \sqrt{\rho}\mathbf{u}, \sqrt{\rho} v)$ satisfies :

$$\left\{ \begin{array}{l}
\partial_t \rho + \mathrm{div}_x(\sqrt{\rho}\sqrt{\rho}\mathbf{u}) + \partial_y(\sqrt{\rho}\sqrt{\rho} v) = 0, \\
\rho_{t=0} = \rho_0.
\end{array} \right.$$

The 3D-CPEs

We define, for any smooth test function φ with compact support such as $\varphi(T,x,y)=0$ and $\varphi_0=\varphi_{t=0}$, the operators :

$$\begin{aligned}
\mathcal{A}(\rho,\mathbf{u},v;\varphi,dy) &= -\int_0^T\int_\Omega \rho\mathbf{u}\partial_t\varphi\,dxdydt \\
&+ \int_0^T\int_\Omega (2\nu_1(t,x,y)\rho D_x(\mathbf{u})-\rho\mathbf{u}\otimes\mathbf{u}):\nabla_x\varphi\,dxdydt \\
&+ \int_0^T\int_\Omega r\rho|\mathbf{u}|\mathbf{u}\varphi\,dxdydt - \int_0^T\int_\Omega \rho\mathrm{div}(\varphi)\,dxdydt \\
&- \int_0^T\int_\Omega \mathbf{u}\partial_y(\nu_2(t,x,y)\partial_y\varphi)\,dxdydt \\
&- \int_0^T\int_\Omega \rho v\mathbf{u}\partial_y\varphi\,dxdydt
\end{aligned}$$

$$\mathcal{B}(\rho,\mathbf{u},v;\varphi,dy)=\int_0^T\int_\Omega \rho v\varphi\,dxdydt$$

and

$$\mathcal{C}(\rho,\mathbf{u};\varphi,dy)=\int_\Omega \rho_{|t=0}\mathbf{u}_{|t=0}\varphi_0\,dxdy$$

A weak solution

Definition

A weak solution of System 3D-CPEs on $[0,T]\times\Omega$, with BC and IC, is a collection of functions $(\rho,\mathbf{u},v)$ such as $\rho\in\mathcal{PE}(\mathbf{u},v;y,\rho_0)$ and the following equality holds for all smooth test function φ with compact support such as $\varphi(T,x,y)=0$ and $\varphi_0=\varphi_{t=0}$:

$$\mathcal{A}(\rho,\mathbf{u},v;\varphi,dy)+\mathcal{B}(\rho,\mathbf{u},v;\varphi,dy)=\mathcal{C}(\rho,\mathbf{u};\varphi,dy)\ .$$

M. Ersoy, T. Ngom, M. Sy
Compressible primitive equations : formal derivation and stability of weak solutions.
submitted to NonLinearity, 2010.

A weak solution

Theorem ([ENS2010])

Let $(\rho_n, \mathbf{u}_n, v_n)$ *be a sequence of weak solutions of System 3D-CPEs, with BC and IC, satisfying an entropy and energy inequality (EEI) such as*

$$\rho_n \geqslant 0, \quad \rho_0^n \to \rho_0 \ in\ L^1(\Omega), \quad \rho_0^n \mathbf{u}_0^n \to \rho_0 \mathbf{u}_0 \ in\ L^1(\Omega).$$

Then, up to a subsequence,

- ρ_n *converges strongly in* $\mathcal{C}^0(0, T; L^{3/2}(\Omega))$,
- $\sqrt{\rho_n}\mathbf{u}_n$ *converges strongly in* $L^2(0, T; (L^{3/2}(\Omega))^2)$,
- $\rho_n u_n$ *converges strongly in* $L^1(0, T; (L^1(\Omega))^2)$ *for all* $T > 0$,
- $(\rho_n, \sqrt{\rho_n}\mathbf{u}_n, \sqrt{\rho_n} v_n)$ *converges to a weak solution of System 3D-CPEs,*
- $(\rho_n, \mathbf{u}_n, v_n)$ *satisfies the EEI and converges to a weak solution of 3D-CPEs-BC.*

M. Ersoy, T. Ngom, M. Sy
Compressible primitive equations : formal derivation and stability of weak solutions.
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Sketch of the proof-step 1

Prove first the stability for the IM' with IC and BC',

$$\begin{cases} \partial_t \xi + \mathrm{div}_x\,(\xi\,\mathbf{u}) + \partial_z\,(\xi\,w) = 0, \\ \partial_t\,(\xi\,\mathbf{u}) + \mathrm{div}_x\,(\xi\,\mathbf{u}\otimes\mathbf{u}) + \partial_z\,(\xi\,\mathbf{u}\,w) + \nabla_x \xi + r\xi|\mathbf{u}|\mathbf{u} = \\ \qquad\qquad 2\bar{\nu}_1 \mathrm{div}_x\,(\xi D_x(\mathbf{u})) + \bar{\nu}_2 \partial_z(\xi \partial_z \mathbf{u}), \\ \partial_z \xi = 0 \end{cases}$$

and by the reverse change of variables “transport” the result to the 3D-CPEs.

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and by the reverse change of variables "transport" the result to the 3D-CPEs. So,

Definition

A weak solution of System IM' on $[0,T]\times\Omega^{'}$, with BC' and IC, is a collection of functions $(\xi,\mathbf{u},w)$, if $\xi\in\mathcal{PE}(\mathbf{u},w;z,\xi_0)$ and the following equality holds for all smooth test function φ with compact support such as $\varphi(T,x,y)=0$ and $\varphi_0=\varphi_{t=0}$:

$$\mathcal{A}(\xi,\mathbf{u},w;\varphi,dz) = \mathcal{C}(\xi,\mathbf{u};\varphi,dz).$$

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$$\mathcal{A}(\xi,\mathbf{u},w;\varphi,dz) = \mathcal{C}(\xi,\mathbf{u};\varphi,dz).$$

Difficulty : show that under suitable sequence of weak solutions, we can pass to the limit in the non-linear term $\xi\,\mathbf{u}\otimes\mathbf{u}$: typically $\sqrt{\xi}\mathbf{u}$ requires strong convergence.

Theorem

Let $(\xi_n, \mathbf{u}_n, w_n)$ be a sequence of weak solutions of the IM' with BC' and IC satisfying an energy and entropy inequality (EEI) such as

$$\xi_n \geqslant 0, \quad \xi_0^n \to \xi_0 \ in\ L^1(\Omega^{'}), \quad \xi_0^n \mathbf{u}_0^n \to \xi_0 \mathbf{u}_0 \ in\ L^1(\Omega^{'}).$$

Then, up to a subsequence,

- ξ_n *converges strongly in* $\mathcal{C}^0(0,T;L^{3/2}(\Omega^{'}))$,
- $\sqrt{\xi_n}\mathbf{u}_n$ *converges strongly in* $L^2(0,T;(L^{3/2}(\Omega^{'}))^2)$,
- $\xi_n u_n$ *converges strongly in* $L^1(0,T;(L^1(\Omega^{'}))^2)$ *for all* $T > 0$,
- $(\xi_n, \sqrt{\xi_n}\mathbf{u}_n, \sqrt{\xi_n}w_n)$ *converges to a weak solution of the IM',*
- $(\xi_n, \mathbf{u}_n, w_n)$ *satisfies the EEI and converges to a weak solution of the IM' with BC'.*

The energy inequality :

$$\frac{d}{dt}\int_{\Omega'} \left(\xi\frac{\mathbf{u}^2}{2} + (\xi\ln\xi - \xi + 1)\right) dxdz + \int_{\Omega'} \xi(2\bar{\nu}_1|D_x(\mathbf{u})|^2 + \bar{\nu}_2|\partial_z\mathbf{u}|^2)\, dxdz + r\int_{\Omega'} \xi|\mathbf{u}|^3\, dxdz \leqslant 0$$

Theorem

Let $(\xi_n, \mathbf{u}_n, w_n)$ be a sequence of weak solutions of the IM' with BC' and IC satisfying an energy and entropy inequality (EEI) such as

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- $(\xi_n, \sqrt{\xi_n}\mathbf{u}_n, \sqrt{\xi_n} w_n)$ *converges to a weak solution of the IM',*
- $(\xi_n, \mathbf{u}_n, w_n)$ *satisfies the EEI and converges to a weak solution of the IM' with BC'.*

The entropy inequality :

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega'} \left(\xi|\mathbf{u} + 2\bar{\nu}_1 \nabla_x \ln \xi|^2 + 2(\xi \log \xi - \xi + 1)\right)\, dxdz \\ &+ \int_{\Omega'} 2\bar{\nu}_1 \xi |\partial_z w|^2 + 2\bar{\nu}_1 \xi |A_x(u)|^2 + \bar{\nu}_2 \xi |\partial_z \mathbf{u}|^2 \, dxdz + \int_{\Omega'} r\xi|\mathbf{u}|^3 + 2\bar{\nu}_1 r |\mathbf{u}|\mathbf{u}\nabla_x \xi \, dxdz \\ &+ \int_{\Omega'} 8\bar{\nu}_1 |\nabla_x \sqrt{\xi}|^2 \, dxdz = 0. \end{aligned}$$

Sketch of the proof-step 2

To prove the stability result on IM', we proceed as follows :

1. we obtain suitable *a priori* bounds on $(\xi, \mathbf{u}, w)$,
   1. we get estimates from the energy inequality,
   2. we get estimates from the BD-entropy inequality, i.e. : a kind of energy with the mulipliер $\mathbf{u} + 2\bar{\nu}_1 \nabla_x \xi$.
2. we show the compactness of sequences $(\xi_n, \mathbf{u}_n, w_n)$ in appropriate space function,
   1. we show the convergence of the sequence $\sqrt{\xi_n}$,
   2. we seek bounds of $\sqrt{\xi_n}\mathbf{u}_n$ and $\sqrt{\xi_n} w_n$,
   3. we prove the convergence of $\xi_n \mathbf{u}_n$,
   4. we prove the convergence of $\sqrt{\xi_n}\mathbf{u}_n$.
3. we prove that we can pass to the limit in all terms of the IM',
4. We “transport” this result with the reverse change of variable to the 3D-CPEs. $\square$

1. Introduction
   - Atmosphere dynamic
   - Sedimentation
   - Unsteady mixed flows in closed water pipes
2. Mathematical results on CPEs
   - An intermediate model
   - Toward an existence result for the 2D-CPEs
   - Toward a stability result for the 3D-CPEs
   - Perspectives
3. An upwinded kinetic scheme for the PFS equations
   - Finite Volume method
   - Kinetic Formulation and numerical scheme
   - Numerical results
   - Perspectives
4. Formal derivation of a SVEs like model
   - A nice coupling : Vlasov and Anisotropic Navier-Stokes equations
   - Hydrodynamic limit, toward a "mixed model"
   - A Viscous Saint-Venant-Exner like model
   - Perspective

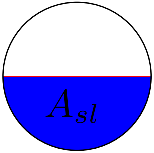
1. Prove the existence of weak solutions of the 3D-CPEs
2. Generalize to any anisotropic pair of viscosities
3. Deal with the case of $p = k\rho^\gamma$, $\gamma \neq 1$, $k = cte$ (also the case $k = k(t, x, y)$)

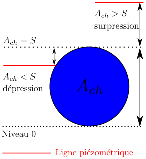
Outline

The PFS Equation are :

$$\left\{\begin{array}{ll}\partial_t(A)+\partial_x(Q) & =0\\[2ex] \partial_t(Q)+\partial_x\left(\dfrac{Q^2}{A}+p(x,A,E)\right) & =-g\,A\,\dfrac{d}{dx}Z(x)\\ & +Pr(x,A,E)\\ & -G(x,A,E)\\ & -g\,K(x,\mathbf{S})\,\dfrac{Q|Q|}{A}\end{array}\right.$$

with $A=\left\{\begin{array}{lll}A_{fs} & \text{if} & \text{FS}\\ A_p & \text{if} & \text{P}\end{array}\right.$





Outline

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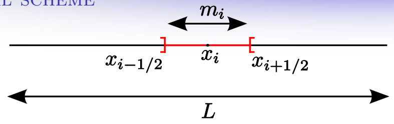
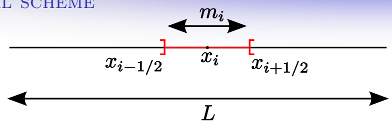


PFS equations under vectorial form :

$$\partial_t \mathbf{U}(t,x) + \partial_x F(x,\mathbf{U}) = \mathcal{S}(t,x)$$

Finite Volume (VF) numerical scheme of order 1

Cell-centered numerical scheme



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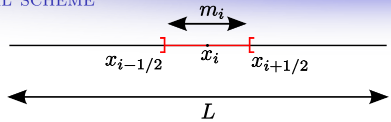

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Cell-centered numerical scheme :

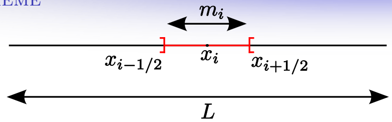
$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right) + \Delta t^n \mathcal{S}(\mathbf{U}_i^n)$$

where

$$\Delta t^n \mathcal{S}_i^n \approx \int_{t_n}^{t_{n+1}} \int_{m_i} \mathcal{S}(t,x)\,dx\,dt$$

Finite Volume (VF) numerical scheme of order 1

Upwinded numerical scheme



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Upwinded numerical scheme :

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x}\left(\widetilde{\mathcal{F}}_{i+1/2} - \widetilde{\mathcal{F}}_{i-1/2}\right)$$

Choice of the numerical fluxes

Our goal : define $\mathcal{F}_{i+1/2}$ in order to preserve continuous properties of the PFS-model

Positivity of A ,

conservativity of A, discrete equilibrium, discrete entropy inequality

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Our choice

VFRoe solver[BEGVF] Kinetic solver[BEG10]

C. Bourdarias, M. Ersoy and S. Gerbi.
A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.
International Journal On Finite Volumes , Vol 6(2) 1–47, 2009.

C. Bourdarias, M. Ersoy and S. Gerbi.
A kinetic scheme for transient mixed flows in non uniform closed pipes : a global manner to upwind all the source terms.
To appear in J. Sci. Comp., 2010.

Outline

We introduce

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1 \ ,$$

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then we define the Gibbs equilibrium by

$$\mathcal{M}(t,x,\xi) = \frac{A(t,x)}{b(t,x)} \chi\left(\frac{\xi - u(t,x)}{b(t,x)}\right)$$

with

$$b(t,x) = \sqrt{\frac{p(t,x)}{A(t,x)}}$$

Since

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1 \ ,$$

and

$$\mathcal{M}(t,x,\xi) = \frac{A(t,x)}{b(t,x)} \, \chi\left(\frac{\xi - u(t,x)}{b(t,x)}\right)$$

then

$$\begin{aligned} A &= \int_{\mathbb{R}} \mathcal{M}(t,x,\xi)\, d\xi \\ Q &= \int_{\mathbb{R}} \xi \mathcal{M}(t,x,\xi)\, d\xi \\ \frac{Q^2}{A} + \underbrace{A\, b^2}_{p} &= \int_{\mathbb{R}} \xi^2 \mathcal{M}(t,x,\xi)\, d\xi \end{aligned}$$

Principle [P02]

The kinetic formulation

(A, Q) is solution of the PFS system if and only if $\mathcal{M}$ satisfy the transport equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\Phi\, \partial_\xi \mathcal{M} = \mathcal{K}(t, x, \xi)$$

where $\mathcal{K}(t, x, \xi)$ is a collision kernel satisfying a.e. (t, x)

$$\int_{\mathbb{R}} \mathcal{K}\, d\xi = 0\ , \ \int_{\mathbb{R}} \xi\, \mathcal{K} d\, \xi = 0$$

and Φ are the source terms.

B. Perthame.
Kinetic formulation of conservation laws.
Oxford University Press.
Oxford Lecture Series in Mathematics and its Applications, Vol 21, 2002.

Principe

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General form of the source terms :

$$\Phi = \overbrace{\frac{d}{dx} Z}^{\text{conservative}} + \overbrace{\mathbf{B} \cdot \frac{d}{dx} \mathbf{W}}^{\text{non conservative}} + \overbrace{K \frac{Q|Q|}{A^2}}^{\text{friction}}$$

with $\mathbf{W} = (Z, S, \cos\theta)$

Discretization of source terms

- Recalling that A, Q and $Z, S, \cos\theta$ constant per mesh
- forgetting the friction : « splitting »...

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Then $\forall (t,x) \in [t_n, t_{n+1}[\times \overset{\circ}{m}_i$

$$\Phi(t,x) = 0$$

since

$$\Phi = \frac{d}{dx} Z + \mathbf{B} \cdot \frac{d}{dx} \mathbf{W}$$

Simplification of the transport equation

- Recalling that A, Q and $Z, S, \cos\theta$ constant per mesh
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$\Longrightarrow$

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} = \mathcal{K}(t,x,\xi)$$

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$\Longrightarrow$

$$\left\{ \begin{array}{rcl} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}(t_n, x, \xi) \overset{def}{:=} \dfrac{A(t_n, x, \xi)}{b(t_n, x, \xi)} \chi \left(\dfrac{\xi - u(t_n, x, \xi)}{b(t_n, x, \xi)} \right) \end{array} \right.$$

by neglecting the collision kernel

Discretization of source terms

On $[t_n, t_{n+1}[\times m_i$, we have :

$$\left\{\begin{array}{lcl} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}_i^n(\xi) \end{array}\right.$$

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i.e.

$$f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \xi \frac{\Delta t^n}{\Delta x} \left(\mathcal{M}_{i+\frac{1}{2}}^-(\xi) - \mathcal{M}_{i-\frac{1}{2}}^+(\xi) \right)$$

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where

$$\mathbf{U}_i^{n+1} = \begin{pmatrix} A_i^{n+1} \\ Q_i^{n+1} \end{pmatrix} \overset{def}{:=} \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f_i^{n+1}(\xi)\, d\xi$$

Discretization of source terms

On $[t_n, t_{n+1}[\times m_i$, we have :

$$\left\{\begin{array}{lcl} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}_i^n(\xi) \end{array}\right.$$

i.e.

$$f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \xi \frac{\Delta t^n}{\Delta x}\left(\mathcal{M}^-_{i+\frac{1}{2}}(\xi) - \mathcal{M}^+_{i-\frac{1}{2}}(\xi)\right)$$

or

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x}\left(\widetilde{\mathcal{F}}^-_{i+1/2} - \widetilde{\mathcal{F}}^+_{i-1/2}\right)$$

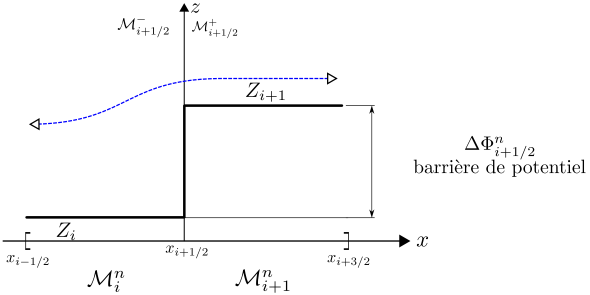
with

$$\widetilde{\mathcal{F}}^{\pm}_{i\pm\frac{1}{2}} = \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}^{\pm}_{i\pm\frac{1}{2}}(\xi)\, d\xi.$$

The microscopic fluxes

Interpretation : potential bareer

$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}^n_i(\xi)}^{\text{positive transmission}}$$
$$+\underbrace{\mathbb{1}_{\{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}>0\}}\mathcal{M}^n_{i+1}\left(-\sqrt{\xi^2-2g\Delta\Phi^n_{i+1/2}}\right)}_{\text{negative transmission}}$$



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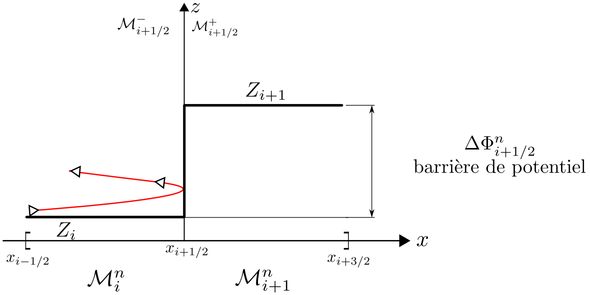
$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}^n_i(\xi)}^{\text{positive transmission}} + \overbrace{\mathbb{1}_{\{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}<0\}}\mathcal{M}^n_i(-\xi)}^{\text{reflection}}$$
$$+ \underbrace{\mathbb{1}_{\{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}>0\}}\mathcal{M}^n_{i+1}\left(-\sqrt{\xi^2-2g\Delta\Phi^n_{i+1/2}}\right)}_{\text{negative transmission}}$$



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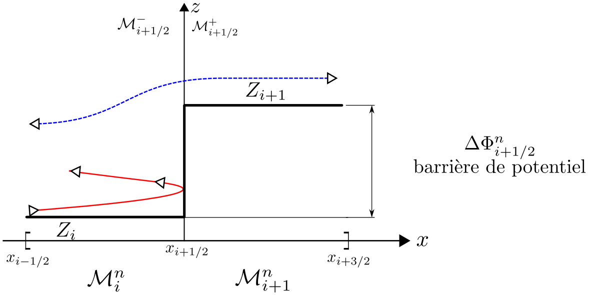

$\Delta\Phi^n_{i+1/2}$ may be interpreted as a time-dependant slope !

The microscopic fluxes

Interpretation : pente dynamique $\Longrightarrow$ décentrement de la friction

$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}^n_i(\xi)}^{\text{positive transmission}} + \overbrace{\mathbb{1}_{\{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}<0\}}\mathcal{M}^n_i(-\xi)}^{\text{reflection}}$$
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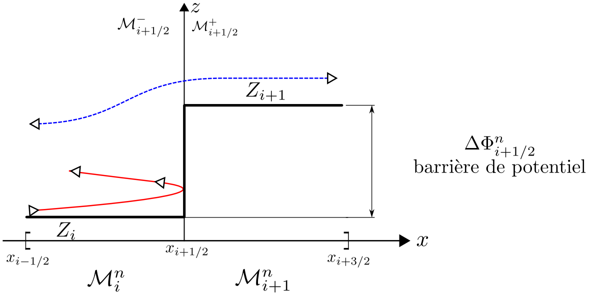

$\Delta\Phi^n_{i+1/2}$ may be interpreted as a time-dependant slope !

. . . we reintegrate the friction . . .

Upwinding of the source terms

- conservative $\partial_x \mathbf{W}$:

$$\mathbf{W}_{i+1} - \mathbf{W}_i$$

- non-conservative $\mathbf{B}\partial_x \mathbf{W}$:

$$\overline{\mathbf{B}}(\mathbf{W}_{i+1} - \mathbf{W}_i)$$

where

$$\overline{\mathbf{B}} = \int_0^1 \mathbf{B}(s, \phi(s, \mathbf{W}_i, \mathbf{W}_{i+1}))\, ds$$

for the « straight lines paths », i.e.

$$\phi(s, \mathbf{W}_i, \mathbf{W}_{i+1}) = s\mathbf{W}_{i+1} + (1-s)\mathbf{W}_i,\ s \in [0,1]$$

G. Dal Maso, P. G. Lefloch and F. Murat.
Definition and weak stability of nonconservative products.
J. Math. Pures Appl. , Vol 74(6) 483–548, 1995.

Numerical properties

With [ABP00]

$$\chi(\omega) = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(\omega)$$

we have :

- Positivity of A (under a CFL condition),
- Conservativity of A,
- Natural treatment of drying and flooding area.

for example

E. Audusse and M-0. Bristeau and B. Perthame.
Kinetic schemes for Saint-Venant equations with source terms on unstructured grids.
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$\longrightarrow$ non well-balanced scheme with such a χ

$\longrightarrow$ but easy computation of the numerical fluxes

E. Audusse and M-0. Bristeau and B. Perthame.
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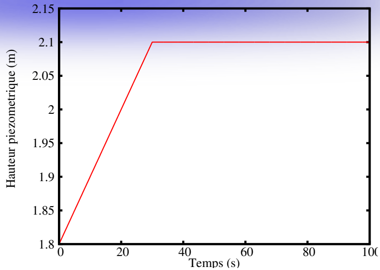
OUTLINE

1. INTRODUCTION
   - Atmosphere dynamic
   - Sedimentation
   - Unsteady mixed flows in closed water pipes
2. MATHEMATICAL RESULTS ON CPES
   - An intermediate model
   - Toward an existence result for the 2D-CPEs
   - Toward a stability result for the 3D-CPEs
   - Perspectives
3. AN UPWINDED KINETIC SCHEME FOR THE PFS EQUATIONS
   - Finite Volume method
   - Kinetic Formulation and numerical scheme
   - **Numerical results**
   - Perspectives
4. FORMAL DERIVATION OF A SVES LIKE MODEL
   - A nice coupling : Vlasov and Anisotropic Navier-Stokes equations
   - Hydrodynamic limit, toward a "mixed model"
   - A Viscous Saint-Venant-Exner like model
   - Perspective

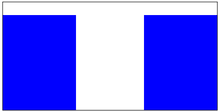
Upwinding of the friction

The « double dam break »

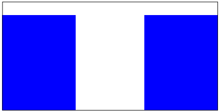
- horizontal pipe : $L = 100\ m$, $R = 1\ m$.
- initial state : $Q = 0\ m^3/s$, $y = 1.8\ m$.
- Symmetric boundary conditions :



downstream and upstream



Décentré $K_s = 1/100$



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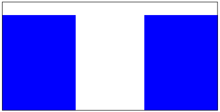
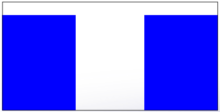
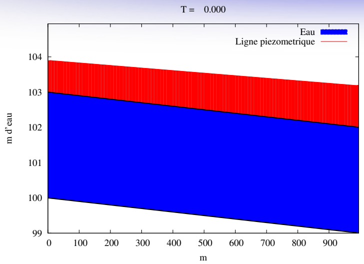


Décentré $K_s = 1/10$

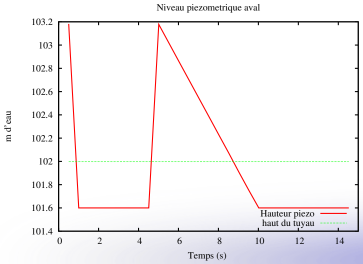


Centré $K_s = 1/10$

Qualitative analysis of convergence



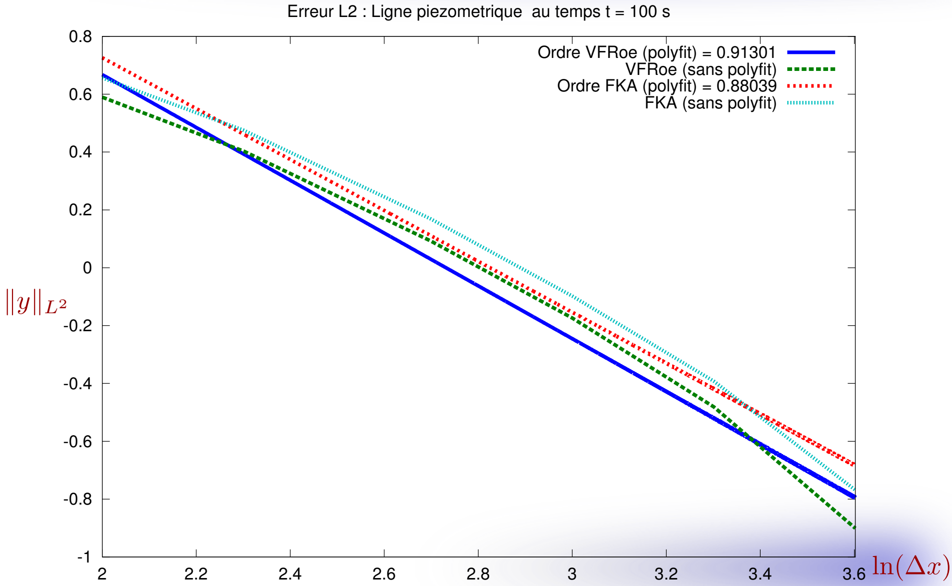
- upstream piezometric head $104\ m$



- downstream piezometric head :

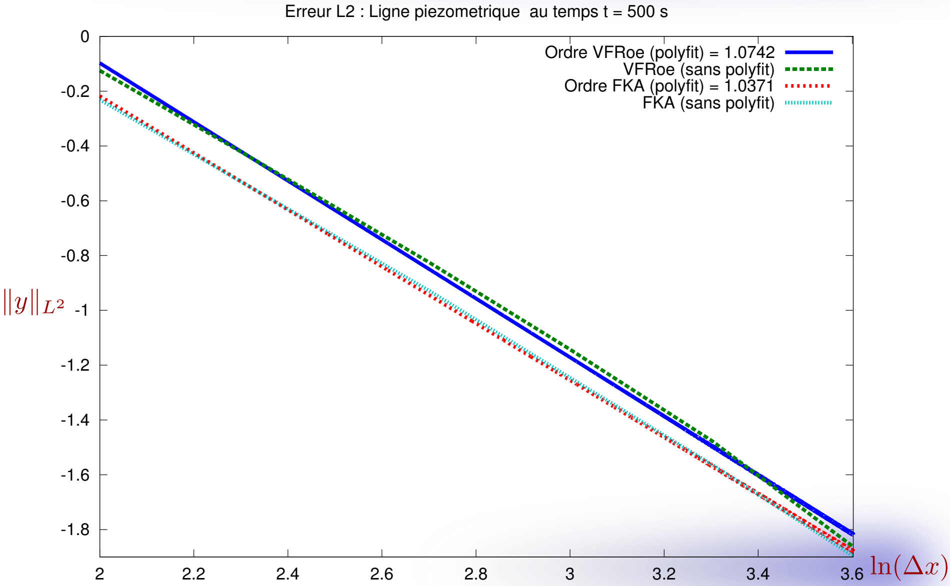
Convergence

During unsteady flows $t = 100\ s$



CONVERGENCE

Stationary $t = 500\ s$



1. Study of the convergence with respect to the χ function
2. Study of the convergence with respect to the paths used to define the non-conservative product

OUTLINE

A Saint-Venant-Exner model

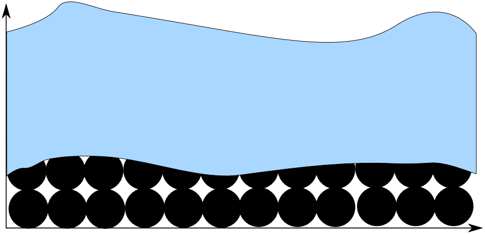
Saint-Venant equations for the hydrodynamic part :

$$\begin{cases} \partial_t h + \mathrm{div}(q) = 0, \\ \partial_t q + \mathrm{div}\left(\dfrac{q \otimes q}{h}\right) + \nabla\left(g\dfrac{h^2}{2}\right) = -gh\nabla b \end{cases}$$

$+$

a bedload transport equation for the morphodynamic part :

$$\partial_t b + \xi \mathrm{div}(q_b(h,q)) = 0$$



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$$\partial_t b + \xi\,\text{div}(q_b(h,q)) = 0$$

with

- h : water height,
- $q = hu$: water discharge,
- q_b : sediment discharge (empirical law : [MPM48], [G81]),
- $\xi = 1/(1-\psi)$: porosity coefficient.

E. Meyer-Peter and R. Müller,
Formula for bed-load transport,
Rep. 2nd Meet. Int. Assoc. Hydraul. Struct. Res., 39–64, 1948.

A.J. Grass,
Sediment transport by waves and currents,
SERC London Cent. Mar. Technol. Report No. FL29 , 1981.

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- $\xi = 1/(1-\psi)$: porosity coefficient.

Our goal : derive formally this type of equation from a non classical way

OUTLINE

The morphodynamic part

is governed by the Vlasov equation :

$$\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v((F + \vec{g})f) = r\Delta_v f$$

where :

- $f(t, x, v)$ density function of particles
- $\vec{g} = (0, 0, -g)^t$,
- $F = \dfrac{6\pi\mu a}{M}(u - v)$ Stokes drag force with
  - a radius of a particle (assumed constant)
  - $M = \rho_p \dfrac{4}{3}\pi a^3$ mass of a particle (assumed constant) with ρ_p density of a particle (assumed constant)
- u fluid velocity
- μ characteristic viscosity of the fluid (assumed constant)
- $r\Delta_v f$ brownian motion of particles where r is the velocity diffusivity

The hydrodynamic part

is governed by the Compressible Navier-Stokes equations

$$\left\{\begin{array}{l} \partial_t \rho_w + \mathsf{div}(\rho_w u) = 0, , \\ \partial_t(\rho_w u) + \mathsf{div}(\rho_w u \otimes u) = \mathsf{div}\sigma(\rho_w, u) + \mathfrak{F}, \\ p = p(t,x) \end{array}\right. \tag{1}$$

where $\sigma(\rho_w, u)$ is the anisotropic total stress tensor :

$$-pI_3 + 2\Sigma(\rho_w).D(u) + \lambda(\rho_w)\mathsf{div}(u)\, I_3$$

The matrix $\Sigma(\rho_w)$ is anisotropic

$$\begin{pmatrix} \mu_1(\rho_w) & \mu_1(\rho_w) & \mu_2(\rho_w) \\ \mu_1(\rho_w) & \mu_1(\rho_w) & \mu_2(\rho_w) \\ \mu_3(\rho_w) & \mu_3(\rho_w) & \mu_3(\rho_w) \end{pmatrix}$$

with $\mu_i \neq \mu_j$ for $i \neq j$ and $i, j = 1, 2, 3$.

The coupling

As the medium may be heterogeneous, we propose the following inhomogeneous pressure law as :

$$p(t,x) = k(t,x_1,x_2)\rho(t,x)^2 \quad \text{with} \quad k(t,x_1,x_2) = \frac{gh(t,x_1,x_2)}{4\rho_f}$$

where $\rho := \rho_w + \rho_s$ is called mixed density
We set ρ_s, the macroscopic density of sediments :

$$\rho_s = \int_{\mathbb{R}^3} f\, dv$$

The last term $\mathfrak{F}$ on the right hand side of CNEs is the effect of the particles motion on the fluid obtained by summing the contribution of all particles :

$$\mathfrak{F} = -\int_{\mathbb{R}^3} F f dv + \rho_w \vec{g} = \frac{9\mu}{2a^2\rho_p} \int_{\mathbb{R}^3} (v-u) f dv + \rho_w \vec{g}.$$

Boundary conditions

- For the hydrodynamic part :
  - on the free surface : a normal stress continuity condition
  - at the movable bottom : a wall-law condition and continuity of the velocity at the interface $x_3 = b(t, \mathbf{x})$

Boundary conditions

- For the hydrodynamic part :
  - on the free surface : a normal stress continuity condition
  - at the movable bottom : a wall-law condition and continuity of the velocity at the interface $x_3 = b(t, \mathbf{x})$
- For the morphodynamic part :
  - kinetic boundary conditions ? (work in progress) replaced by the equation :

$$S = \partial_t b + \sqrt{1 + |\nabla_{\mathbf{x}} b|^2} u_{|x_3=b} \cdot n_b$$

and $S - \sqrt{1 + |\nabla_{\mathbf{x}} b|^2} u_{|x_3=b} \cdot n_b$ takes into account incoming and outgoing particles.

Outline

Rescaling for both models, "set $\varepsilon = 0$"

Let

- $\sqrt{\theta}$ be the fluctuation of kinetic velocity,
- $\mathfrak{U}$ be a characteristic vertical velocity of the fluid,
- $\mathfrak{T}$ be a characteristic time,
- τ be a relaxation time,
- $\mathfrak{L}$ be a characteristic vertical height,

and

$$B = \frac{\sqrt{\theta}}{\mathfrak{U}}, \quad C = \frac{\mathfrak{T}}{\tau}, \quad F = \frac{g\mathfrak{T}}{\sqrt{\theta}}, \quad E = \frac{2}{9}\left(\frac{a}{\mathfrak{L}}\right)^2 \frac{\rho_p}{\rho_f} C$$

with the following asymptotic regime :

$$B = O(1), \quad C = \frac{1}{\varepsilon}, \quad F = O(1), \quad E = O(1).$$

T. Goudon and P-E. Jabin and A. Vasseur,
Hydrodynamic limit for the Vlasov-Navier-Stokes Equations. I. Light particles regime,
Indiana Univ. Math. J., 53(6) :1495–1515,2004.

THE "MIXED" MODEL :

Formally, $\varepsilon \to 0$, we obtain :

- Takes the two first moments of the the hydrodynamic limit of Vlasov equation

$$+$$

- Rescaled Navier Stokes Equation

$$=$$

$$\left\{\begin{array}{l}\partial_t\rho + \mathrm{div}(\rho u) = 0,\\ \partial_t(\rho\mathbf{u}) + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}\otimes\mathbf{u}) + \partial_{x_3}(\rho\mathbf{u}v) + \nabla_{\mathbf{x}}P\\ = \mathrm{div}_{\mathbf{x}}\left(\mu_1(\rho)D_{\mathbf{x}}(\mathbf{u})\right) + \partial_{x_3}\Big(\mu_2(\rho)(\partial_{x_3}\mathbf{u} + \nabla_{\mathbf{x}}u_3)\Big)\\ +\nabla_x(\lambda(\rho)\mathrm{div}(u))\\ \\ \partial_t(\rho u_3) + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}u_3) + \partial_{x_3}(\rho u_3^2) + \partial_{x_3}P\\ = \mathrm{div}_{\mathbf{x}}\Big(\mu_2(\rho)(\partial_{x_3}\mathbf{u} + \nabla_{\mathbf{x}}u_3)\Big) + \partial_{x_3}(\mu_3(\rho)\partial_{x_3}u_3)\\ +\partial_{x_3}(\lambda(\rho)\mathrm{div}(u))\end{array}\right.$$

where

$$P = p + \theta\rho_s$$

and

$$\rho = \rho_w + \rho_s.$$

Outline

Applying an asymptotic analysis to the mixed model :
we finally obtain :

$$\partial_t(h\bar{u}) + \mathsf{div}(h\bar{u}\otimes\bar{u}) + \frac{1}{3\,F_r^2}\nabla h^2 = -\frac{h}{F_r^2}\nabla b + \mathsf{div}(hD(\bar{u})) - \begin{pmatrix} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u) \end{pmatrix}$$

$$S = \partial_t b + \sqrt{1+|\nabla_{\mathsf{x}} b|^2}\, u_{|x_3=b}\cdot n_b$$

- find appropriate kinematic boundary condition
- generalize this procedure to a real mixed model
- justify such a formal derivation mathematically

Thank you for attention