Modeling, Mathematical and Numerical Analysis for some Compressible and Incompressible Equations in Thin Layer.

M. Ersoy

15 October 2010
1 Introduction
- Atmosphere dynamic
- Sedimentation
- Unsteady mixed flows in closed water pipes

2 Mathematical results on CPEs
- An intermediate model
- Toward an existence result for the 2D-CPEs
- Toward a stability result for the 3D-CPEs
- Perspectives

3 An upwinded kinetic scheme for the PFS equations
- Finite Volume method
- Kinetic Formulation and numerical scheme
- Numerical results
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4 Formal derivation of a SVEs like model
- A nice coupling: Vlasov and Anisotropic Navier-Stokes equations
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Hydrostatic approximation and averaged equations

Navier Stokes equations (NSEs) or Euler equations (EEs) on
\( \Omega = \{(x, y) \in \mathbb{R}^3; H \ll L\} \) "thin layer domain"
Hydrostatic approximation and averaged equations

Navier Stokes equations (NSEs) or Euler equations (EEs) on
\[ \Omega = \{(x, y) \in \mathbb{R}^3; H \ll L\} \] "thin layer domain"

↓ [Ped]

Hydrostatic approximation (asymptotic analysis with \( \varepsilon = H/L = W/V \ll 1 \) and
rescaling \( \tilde{x} = x/L, \tilde{y} = y/H, \tilde{u} = u/U \tilde{w} = w/W \) ) \( \rightarrow \) Primitive equations (PEs)

---

J. Pedloski
Geophysical Fluid Dynamics.
Hydrostatic approximation and averaged equations

Navier Stokes equations (NSEs) or Euler equations (EEs) on 

\[ \Omega = \{(x, y) \in \mathbb{R}^3; H \ll L\} \] "thin layer domain"

\[ \downarrow \text{[Ped]} \]

Hydrostatic approximation (asymptotic analysis with \( \varepsilon = H/L = W/V \ll 1 \) and rescaling \( \tilde{x} = x/L, \tilde{y} = y/H, \tilde{u} = u/U \tilde{w} = w/W \) ) \( \rightarrow \) Primitive equations (PEs)

\[ \downarrow \text{[GP]} \]

Averaged PEs with respect to depth or altitude \( y \) \( \rightarrow \) Saint-Venant Equations (SVEs)

---

J. Pedlosk

Geophysical Fluid Dynamics.


J.-F Gerbeau and B. Perthame

Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation.

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**Atmosphere dynamic**

**Dynamic:**
- Compressible fluid
- Small vertical extension with respect to horizontal
- Principally horizontal movements
- Density highly stratified
Atmosphere dynamic

- **Dynamic:**
  - Compressible fluid
  - Small vertical extension with respect to horizontal
  - Principally horizontal movements
  - Density highly stratified

- **Modeling:** **Compressible** Navier-Stokes equations
  Hydrostatic approximation \(\rightarrow\) compressible primitive equations (CPEs)

\[
\begin{align*}
\partial_t \rho + \text{div}_x(\rho u) + \partial_y(\rho v) &= 0 \\
\partial_t(\rho u) + \text{div}_x(\rho u \otimes u) + \partial_y(\rho uv) + \nabla_x p &= \text{div}_x(\sigma_x) + f \\
\partial_t(\rho v) + \text{div}_x(\rho uv) + \partial_y(\rho v^2) + \partial_y p &= -\rho g + \text{div}_y(\sigma_y) \\
p &= c^2 \rho
\end{align*}
\]
Atmosphere Dynamic

Dynamic:
- Compressible fluid
- Small vertical extension with respect to horizontal
- Principally horizontal movements
- Density highly stratified

Modeling: Compressible Navier-Stokes equations

\[ \begin{align*}
\partial_t \rho + \text{div}_x (\rho \mathbf{u}) + \partial_y (\rho v) &= 0 \\
\partial_t (\rho \mathbf{u}) + \text{div}_x (\rho \mathbf{u} \otimes \mathbf{u}) + \partial_y (\rho \mathbf{u} \mathbf{v}) + \nabla_x p &= \text{div}_x (\sigma_x) + f \\
\partial_t (\rho v) + \text{div}_x (\rho \mathbf{u} v) + \partial_y (\rho v^2) + \partial_y p &= -\rho g + \text{div}_y (\sigma_y) \\
p &= c^2 \rho
\end{align*} \]

Hydrostatic approximation \(\rightarrow\) compressible primitive equations (CPEs)
Atmosphere dynamic

Dynamic:
- Compressible fluid
- Small vertical extension with respect to horizontal
- Principally horizontal movements
- Density highly stratified $p = \xi(t, x)e^{-g/c^2 y}$

Modeling: Compressible Navier-Stokes equations

Hydrostatic approximation $\rightarrow$ compressible primitive equations (CPEs)

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho \mathbf{u}) + \partial_y (\rho v) &= 0 \\
\partial_t (\rho \mathbf{u}) + \text{div}_x (\rho \mathbf{u} \otimes \mathbf{u}) + \partial_y (\rho \mathbf{u} v) + \nabla_x p &= \text{div}_x (\sigma_x) + f \\
\partial_y p &= -\rho g \\
p &= c^2 \rho
\end{align*}
\]
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**Sedimentation**

- **Sediment**: produced by erosion process

- **Dynamic**:
  - Incompressible fluid
  - Small vertical extension with respect to horizontal
  - Principally horizontal movements

**Modeling**: Saint-Venant-Exner equations

- Hydrodynamic part
  - Saint-Venant equations (averaged IPEs)
  \[
  \begin{align*}
  \frac{\partial}{\partial t} h + \text{div}(q) &= 0, \\
  \frac{\partial}{\partial t} q + \text{div}(q \otimes q h) + \nabla (gh^2) &= -gh \nabla b
  \end{align*}
  \]

- Morphodynamic part
  - Exner equations
  \[
  \begin{align*}
  \frac{\partial}{\partial t} b + \xi \text{div}(q b(q,h)) &= 0
  \end{align*}
  \]

M. Ersoy (BCAM)
**Sedimentation**

- **Sediment**: produced by erosion process

- **Dynamic**:
  - Incompressible fluid
  - Small vertical extension with respect to horizontal
  - Principally horizontal movements

- **Modeling**: Saint-Venant-Exner equations
  - Hydrodynamic part $\rightarrow$ Saint-Venant equations (averaged IPEs)
  - Morphodynamic part $\rightarrow$ Exner equations
**SEDIMENTATION**

- **Sediment**: produced by erosion process

- **Dynamic**:
  - Incompressible fluid
  - Small vertical extension with respect to horizontal
  - Principally horizontal movements

- **Modeling**: Saint-Venant-Exner equations
  - hydrodynamic part $\rightarrow$ **Saint-Venant equations** (averaged IPEs)
    \[
    \begin{align*}
    \partial_t h + \text{div}(q) &= 0, \\
    \partial_t q + \text{div} \left( \frac{q \otimes q}{h} \right) + \nabla \left( g \frac{h^2}{2} \right) &= -gh \nabla b
    \end{align*}
    \]
  - morphodynamic part $\rightarrow$ **Exner equations**
**Sedimentation**

- **Sediment**: produced by erosion process

**Dynamic**:  
- Incompressible fluid  
- Small vertical extension with respect to horizontal  
- Principally horizontal movements  
- Variable bottom, example: bed river

**Modeling**: Saint-Venant-Exner equations  
- Hydrodynamic part $\rightarrow$ Saint-Venant equations (averaged IPEs)
  
$$
\begin{align*}
\partial_t h + \text{div}(q) &= 0, \\
\partial_t q + \text{div} \left( \frac{q \otimes q}{h} \right) + \nabla \left( gh^2 \right) &= -gh \nabla b
\end{align*}
$$

- Morphodynamic part $\rightarrow$ Exner equations

$$
\partial_t b + \xi \text{div}(q_b(h, q)) = 0
$$
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Unsteady mixed flows in closed water pipes

- **mixed**: Free surface and pressurized flows
  - Free Surface area (FS)
    - Section non filled and **incompressible** flow...

\[ z = R(x) \]
\[ z = Z(x) \]
\[ z = -R(x) \]

Free surface
\[ \mathcal{H}(A) \]
\[ \Omega(t, x) \]
Unsteady mixed flows in closed water pipes

- **Mixed**: Free surface and pressurized flows
  - Free Surface area (FS)
    - Section non filled and incompressible flow...
  - Pressurized area (P)
    - Section completely filled and *compressible* flow...

\[ z = R(x) \]
\[ z = Z(x) \]
\[ z = -R(x) \]
Unsteady mixed flows in closed water pipes

Dynamic:
- Incompressible or compressible fluid following the area
- Small vertical extension with respect to horizontal
- Principally horizontal movements: unidirectional
Unsteady mixed flows in closed water pipes

Dynamic:
- Incompressible or compressible fluid following the area
- Small vertical extension with respect to horizontal
- Principally horizontal movements: unidirectional

Modeling: A nice coupling of Saint-Venant like equations
- free surface part → usual Saint-Venant equations
- pressurized part → Saint-Venant like equations
Unsteady mixed flows in closed water pipes

- **Dynamic:**
  - Incompressible or compressible fluid following the area
  - Small vertical extension with respect to horizontal
  - Principally horizontal movements: unidirectional

- **Modeling:** A nice coupling of Saint-Venant-like equations
  - Free surface part $\rightarrow$ usual Saint-Venant equations
    \[
    \begin{align*}
    \partial_t A_{fs} + \partial_x Q_{fs} &= 0, \\
    \partial_t Q_{fs} + \partial_x \left( \frac{Q_{fs}^2}{A_{fs}} + p_{fs}(x, A_{fs}) \right) &= -gA_{fs} \frac{dZ}{dx} + Pr_{fs}(x, A_{fs}) - G(x, A_{fs}) - K(x, A_{fs}) \frac{Q_{fs} |Q_{fs}|}{A_{fs}}
    \end{align*}
    \]
  - Pressurized part $\rightarrow$ Saint-Venant-like equations
Unsteady mixed flows in closed water pipes

- **Dynamic**:
  - Incompressible or compressible fluid following the area
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- **Modeling**: A nice coupling of Saint-Venant like equations
  - Free surface part → usual Saint-Venant equations
    \[
    \begin{align*}
    \partial_t A_{fs} + \partial_x Q_{fs} &= 0, \\
    \partial_t Q_{fs} + \partial_x \left( \frac{Q_{fs}^2}{A_{fs}} + p_{fs}(x, A_{fs}) \right) &= -g A_{fs} \frac{d Z}{dx} + Pr_{fs}(x, A_{fs}) - G(x, A_{fs}) - K(x, A_{fs}) \frac{Q_{fs} |Q_{fs}|}{A_{fs}}
    \end{align*}
    \]
  - Pressurized part → Saint-Venant like equations
    \[
    \begin{align*}
    \partial_t A_p + \partial_x Q_p &= 0, \\
    \partial_t Q_p + \partial_x \left( \frac{Q_p^2}{A_p} + p_p(x, A_p) \right) &= -g A_p \frac{d Z}{dx} + Pr_p(x, A_p) - G(x, A_p) - K(x, A_p) \frac{Q_p |Q_p|}{A_p}
    \end{align*}
    \]
Unsteady mixed flows in closed water pipes

**Dynamic:**
- Incompressible or compressible fluid following the area
- Small vertical extension with respect to horizontal
- Principally horizontal movements: unidirectional

**Modeling:** A nice coupling: The PFS model
- from the coupling:

\[
\begin{align*}
A &= \begin{cases} 
A_{fs} & \text{if } FS \\
A_p & \text{if } P
\end{cases} : \text{ the mixed variable} \\
Q &= Au : \text{ the discharge}
\end{align*}
\]

\[
\begin{align*}
\partial_t(A) + \partial_x(Q) &= 0 \\
\partial_t(Q) + \partial_x \left( \frac{Q^2}{A} + p(x, A, E) \right) &= -g A \frac{d}{dx} Z(x) \\
&\quad + Pr(x, A, E) - G(x, A, E) - g K(x, S) \frac{Q|Q|}{A}
\end{align*}
\]

where \( E \) is a state indicator and appropriate \( p \) and \( Pr \)
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Energy estimates?

CPEs:

\[
\begin{aligned}
&\partial_t \rho + \text{div}_x (\rho \, \mathbf{u}) + \partial_y (\rho \, \mathbf{v}) = 0, \\
&\partial_t (\rho \, \mathbf{u}) + \text{div}_x (\rho \, \mathbf{u} \otimes \mathbf{u}) + \partial_y (\rho \, \mathbf{v} \, \mathbf{u}) + \nabla_x p(\rho) = 2\text{div}_x (\nu_1 \text{D}_x (\mathbf{u})) + \partial_y (\nu_2 \partial_y \mathbf{u}), \\
&\partial_y p(\rho) = -g \rho \\
p(\rho) = c^2 \rho
\end{aligned}
\]
**Energy estimates?**

CPEs:

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho \, u) + \partial_y (\rho \, v) &= 0, \\
\partial_t (\rho \, u) + \text{div}_x (\rho \, u \otimes u) + \partial_y (\rho \, v u) + \nabla_x p(\rho) &= 2 \text{div}_x (\nu_1 D_x(u)) + \partial_y (\nu_2 \partial_y u), \\
\partial_y p(\rho) &= -g \rho \\
p(\rho) &= c^2 \rho
\end{align*}
\]

**Problem**: How to obtain energy estimates since: the sign of

\[
\int_\Omega \rho g v \, dx \, dy
\]

\[
\frac{d}{dt} \int_\Omega \rho |u|^2 + \rho \ln \rho - \rho + 1 \, dx \, dy + \int_\Omega 2 \nu_1 |D_x(u)|^2 + \nu_2 |\partial_y u| \, dx \, dy + \int_\Omega \rho g v \, dx \, dy = 0
\]

is unknown!
Energy estimates?

CPEs:

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho \mathbf{u}) + \partial_y (\rho v) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div}_x (\rho \mathbf{u} \otimes \mathbf{u}) + \partial_y (\rho v \mathbf{u}) + \nabla_x p(\rho) &= 2\text{div}_x (\nu_1 D_x (\mathbf{u})) + \partial_y (\nu_2 \partial_y \mathbf{u}), \\
\partial_y p(\rho) &= -g\rho \\
p(\rho) &= c^2 \rho
\end{align*}
\]

Problem: How to obtain energy estimates since: the sign of

\[
\int_\Omega \rho g v \, dx \, dy
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\[
\frac{d}{dt} \int_\Omega \rho |u|^2 + \rho \ln \rho - \rho + 1 \, dx \, dy + \int_\Omega 2\nu_1 |D_x (u)|^2 + \nu_2 |\partial_y^2 u| \, dx \, dy + \int_\Omega \rho g v \, dx \, dy = 0
\]

is unknown!

Consequently, standard techniques fail.
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**The key point: the hydrostatic equation**

Using the hydrostatic equation, we obviously have:

\[ \rho(t, x, y) = \xi(t, x) e^{-g/c^2 y} \]

for some function \( \xi(t, x) \): \( \rho \) is stratified
The key point: the hydrostatic equation

Using the hydrostatic equation, we obviously have:

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Problem: find equations satisfied by \( \xi \)
**The key point : the hydrostatic equation**

Using the hydrostatic equation, we obviously have:

\[ \rho(t, x, y) = \xi(t, x)e^{-g/c^2y} \]

for some function \( \xi(t, x) : \rho \) is stratified

**Problem : find equations satisfied by \( \xi \)**

**An intermediate model :**

- replace \( \rho \) by \( \xi e^{-g/c^2y} \) in CPEs

\[
\begin{align*}
\partial_t (\xi e^{-g/c^2y}) + \text{div}_x \left( \xi e^{-g/c^2y} u \right) + \partial_y \left( \xi e^{-g/c^2y} v \right) &= 0, \\
\partial_t \left( \xi e^{-g/c^2y} u \right) + \text{div}_x \left( \xi e^{-g/c^2y} u \otimes u \right) + \partial_y \left( \xi e^{-g/c^2y} v u \right) + \nabla_x c^2 \nabla_x (\xi e^{-g/c^2y}) &= 2\text{div}_x (\nu_1 D_x (u)) + \partial_y (\nu_2 \partial_y u), \\
\rho &= \xi e^{-g/c^2y}
\end{align*}
\]

- multiply CPEs by \( e^{+g/c^2y} \)
The key point: the hydrostatic equation

Using the hydrostatic equation, we obviously have:

$$\rho(t, x, y) = \xi(t, x)e^{-g/c^2y}$$

for some function $\xi(t, x)$: $\rho$ is stratified

Problem: find equations satisfied by $\xi$

An intermediate model:

- replace $\rho$ by $\xi e^{-g/c^2y}$ in CPEs
- multiply CPEs by $e^{+g/c^2y}$

\[
\begin{align*}
\partial_t(\xi) + \text{div}_x (\xi \mathbf{u}) + e^{g/c^2y} \partial_y (\xi e^{-g/c^2y} v) &= 0, \\
\partial_t (\xi \mathbf{u}) + \text{div}_x (\xi \mathbf{u} \otimes \mathbf{u}) + e^{g/c^2y} \partial_y (\xi e^{-g/c^2y} \mathbf{u}) + c^2 \nabla_x \xi &= 2e^{g/c^2y} \text{div}_x (\nu_1 D_x(u)) + e^{g/c^2y} \partial_y (\nu_2 \partial_y u), \\
\rho &= \xi e^{-g/c^2y}
\end{align*}
\]

- set $z = 1 - e^{-g/c^2y}$ such that $e^{g/c^2y} \partial_y = \partial_z$ and $w = e^{-g/c^2y} v$ under suitable choice of viscosities.
The key point: the hydrostatic equation

Using the hydrostatic equation, we obviously have:

\[ \rho(t, x, y) = \xi(t, x) e^{-g/c^2 y} \]

for some function \( \xi(t, x) : \rho \) is stratified

Problem: find equations satisfied by \( \xi \)

An intermediate model:

\[
\begin{cases}
\partial_t \xi + \text{div}_x (\xi \mathbf{u}) + \xi \partial_z w = 0, \\
\partial_t (\xi \mathbf{u}) + \text{div}_x (\xi \mathbf{u} \otimes \mathbf{u}) + \partial_z (\xi \mathbf{w} \mathbf{u}) + c^2 \nabla_x (\xi) = 2\text{div}_x (\nu_1 D_x (\mathbf{u})) + \partial_z (\nu_2 \partial_z \mathbf{u}), \\
\partial_z \xi = 0
\end{cases}
\]
The key point: the hydrostatic equation

Using the hydrostatic equation, we obviously have:

\[ \rho(t, x, y) = \xi(t, x) e^{-g/c^2 y} \]

for some function \( \xi(t, x) \): \( \rho \) is stratified

Problem: find equations satisfied by \( \xi \)

An intermediate model:

\[
\begin{align*}
\partial_t \xi + \text{div}_x (\xi u) + \xi \partial_z w &= 0, \\
\partial_t (\xi u) + \text{div}_x (\xi u \otimes u) + \partial_z (\xi w u) + c^2 \nabla_x (\xi) &= 2\text{div}_x (\nu_1 D_x (u)) + \partial_z (\nu_2 \partial_z u), \\
\partial_z \xi &= 0
\end{align*}
\]

\[
\frac{d}{dt} \int_{\Omega} \xi |u|^2 + \xi \ln \xi - \xi + 1 \, dx \, dz + \int_{\Omega} 2\nu_1 |D_x (u)|^2 + \nu_2 |\partial^2_z u| \, dx \, dz = 0
\]
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The 2D-CPEs

We set

\[
\begin{align*}
\nu_1(t, x, y) &= \nu_0 e^{-g/c^2 y} \text{ for some given positive constant } \nu_0, \\
\nu_2(t, x, y) &= \nu_1 e^{g/c^2 y} \text{ for some given positive constant } \nu_1.
\end{align*}
\]

the boundary conditions (BC)

\[
\begin{align*}
u\big|_{x=0} &= u\big|_{x=l} = 0, \\
v\big|_{y=0} &= v\big|_{y=h} = 0, \\
\partial_y u\big|_{y=0} &= \partial_y u\big|_{y=h} = 0
\end{align*}
\]

and the initial conditions (IC)

\[
\begin{align*}
u|_{t=0} &= u_0(x, y), \\
p|_{t=0} &= \xi_0(x)e^{-g/c^2 y}
\end{align*}
\]

where \(\xi_0: \)

\[
0 < m \leq \xi_0 \leq M < \infty.
\]
**Theorem ([EN2010])**

Suppose that initial data \((\xi_0, u_0)\) have the properties:

\[(\xi_0, u_0) \in W^{1,2}(\Omega), \quad u_0|_{x=0} = u_0|_{x=l} = 0.\]

Then \(\rho(t, x, y)\) is a bounded strictly positive function and the 2D-CPEs with BC has a weak solution in the following sense: a weak solution of 2D-CPEs with BC is a collection \((\rho, u, v)\) of functions such that \(\rho \geq 0\) and

\[
\rho \in L^\infty(0, T; W^{1,2}(\Omega)), \quad \partial_t \rho \in L^2(0, T; L^2(\Omega)),
\]

\[
u \in L^2(0, T; W^{2,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)), \quad v \in L^2(0, T; L^2(\Omega))
\]

which satisfies the 2D-CPEs in the distribution sense; in particular, the integral identity holds for all \(\phi|_{t=T} = 0\) with compact support:

\[
\int_0^T \int_\Omega \rho u \partial_t \phi + \rho u^2 \partial_x \phi + \rho uv \partial_y \phi + \rho \partial_x \phi + \rho \partial_y \phi \, dx \, dy \, dt
\]

\[
= - \int_0^T \int_\Omega \nu_1 \partial_x u \partial_x \phi + \nu_2 \partial_y u \partial_y \phi \, dx \, dy \, dt + \int_\Omega u_0 \rho_0 \phi|_{t=0} \, dx \, dy
\]
THE PROOF
The intermediate model (IM) is exactly the model studied by Gatapov et al [GK05], derived from Equations 2D-CPEs by neglecting some terms, for which they provide the following global existence result:

**Theorem (B. Gatapov and A.V. Kazhikhov 2005)**

Suppose that initial data \((\xi_0, u_0)\) have the properties:

\[
(\xi_0, u_0) \in W^{1,2}(\Omega), \quad u_0|_{x=0} = u_0|_{x=1} = 0.
\]

Then \(\xi(t, x)\) is a bounded strictly positive function and the IM has a weak solution in the following sense: a weak solution of the IM satisfying the BC is a collection \((\xi, u, w)\) of functions such that \(\xi \geq 0\) and

\[
\xi \in L^\infty(0, T; W^{1,2}(0, 1)), \quad \partial_t \xi \in L^2(0, T; L^2(0, 1)),
\]

\[
u \in L^2(0, T; W^{2,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)), \quad w \in L^2(0, T; L^2(\Omega))
\]

which satisfy the IM in the distribution sense.
THE PROOF

By the simple change of variables $z = 1 - e^{-y}$ in the integrals, we get:

- $\| \rho \|_{L^2(\Omega)} = \alpha \| \xi \|_{L^2(\Omega)}$,
- $\| \nabla_x \rho \|_{L^2(\Omega)} = \alpha \| \nabla_x \xi \|_{L^2(\Omega)}$,
- $\| \partial_y \rho \|_{L^2(\Omega)} = \alpha \| \xi \|_{L^2(\Omega)}$

where $\alpha = \int_0^{1-e^{-1}} (1 - z) \, dz < +\infty$. We deduce then,

$$\| \rho \|_{W^{1,2}(\Omega)} = \alpha \| \xi \|_{W^{1,2}(\Omega)}$$

which provides

$$\rho \in L^\infty(0, T; W^{1,2}(\Omega)) \text{ and } \partial_t \rho \in L^2(0, T; L^2(\Omega)).$$

$v \in L^2(0, T; L^2(\Omega))$ since the inequality holds:

$$\| v \|_{L^2(\Omega)} = \int_0^1 \int_0^1 |v(t, x, y)|^2 \, dy \, dx$$

$$= \int_0^1 \int_0^{1-e^{-1}} \left( \frac{1}{1 - z} \right)^3 |w(t, x, z)|^2 \, dz \, dx$$

$$< e^3 \| w \|_{L^2(\Omega)}.$$

Finally, all estimates on $u$ remain true. □
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The 3D-CPEs

We set
\[ \nu_1(t, x, y) = \tilde{\nu}_1 \rho(t, x, y) \text{ and } \nu_2 = \tilde{\nu}_2 \rho(t, x, y) e^{2y}. \]

for some positive constant \( \tilde{\nu}_1 \) and \( \tilde{\nu}_2 \).

We consider the IC and BC’ where we prescribe periodic conditions on the spatial domain with respect to \( x \).

We define the set of function \( \rho \in \mathcal{PE}(u, v; y, \rho_0) \) such that

\[ \rho \in L^\infty(0, T; L^3(\Omega)), \quad \sqrt{\rho} \in L^\infty(0, T; H^1(\Omega)), \]

\[ \sqrt{\rho}u \in L^2(0, T; (L^2(\Omega))^2), \quad \sqrt{\rho}v \in L^\infty(0, T; L^2(\Omega)), \]

\[ \sqrt{\rho}D_x(u) \in L^2(0, T; (L^2(\Omega))^{2\times2}), \quad \sqrt{\rho} \partial_y v \in L^2(0, T; L^2(\Omega)), \]

\[ \nabla \sqrt{\rho} \in L^2(0, T; (L^2(\Omega))^3) \]

with \( \rho \geq 0 \) and where \((\rho, \sqrt{\rho}u, \sqrt{\rho}v)\) satisfies:

\[
\begin{align*}
\partial_t \rho + \text{div}_x(\sqrt{\rho} \sqrt{\rho}u) + \partial_y(\sqrt{\rho} \sqrt{\rho}v) &= 0, \\
\rho_{t=0} &= \rho_0.
\end{align*}
\]
The 3D-CPEs

We define, for any smooth test function $\varphi$ with compact support such as $\varphi(T, x, y) = 0$ and $\varphi_0 = \varphi_{t=0}$, the operators:

\[ \mathcal{A}(\rho, u, v; \varphi, dy) = - \int_0^T \int_\Omega \rho u \partial_t \varphi \, dx \, dy \, dt \]

\[ + \int_0^T \int_\Omega (2 \nu_1(t, x, y) \rho D_x(u) - \rho u \otimes u) : \nabla_x \varphi \, dx \, dy \, dt \]

\[ + \int_0^T \int_\Omega r \rho |u| u \varphi \, dx \, dy \, dt - \int_0^T \int_\Omega \rho \text{div}(\varphi) \, dx \, dy \, dt \]

\[ - \int_0^T \int_\Omega u \partial_y (\nu_2(t, x, y) \partial_y \varphi) \, dx \, dy \, dt \]

\[ - \int_0^T \int_\Omega \rho v u \partial_y \varphi \, dx \, dy \, dt \]

\[ \mathcal{B}(\rho, u, v; \varphi, dy) = \int_0^T \int_\Omega \rho v \varphi \, dx \, dy \, dt \]

and

\[ \mathcal{C}(\rho, u; \varphi, dy) = \int_\Omega \rho |_{t=0} u |_{t=0} \varphi_0 \, dx \, dy \]
A weak solution of System 3D-CPEs on \([0, T] \times \Omega\), with BC and IC, is a collection of functions \((\rho, u, v)\) such as \(\rho \in \mathcal{PE}(u, v; y, \rho_0)\) and the following equality holds for all smooth test function \(\varphi\) with compact support such as \(\varphi(T, x, y) = 0\) and \(\varphi_0 = \varphi_{t=0}\):

\[
A(\rho, u, v; \varphi, dy) + B(\rho, u, v; \varphi, dy) = C(\rho, u; \varphi, dy).
\]
A weak solution

Theorem ([ENS2010])

Let \((\rho_n, u_n, v_n)\) be a sequence of weak solutions of System 3D-CPEs, with BC and IC, satisfying an entropy and energy inequality (EEI) such as

\[
\rho_n \geq 0, \quad \rho^n_0 \to \rho_0 \text{ in } L^1(\Omega), \quad \rho^n_0 u^n_0 \to \rho_0 u_0 \text{ in } L^1(\Omega).
\]

Then, up to a subsequence,

- \(\rho_n\) converges strongly in \(C^0(0, T; L^{3/2}(\Omega))\),
- \(\sqrt{\rho_n u_n}\) converges strongly in \(L^2(0, T; (L^{3/2}(\Omega))^2)\),
- \(\rho_n u_n\) converges strongly in \(L^1(0, T; (L^1(\Omega))^2)\) for all \(T > 0\),
- \((\rho_n, \sqrt{\rho_n u_n}, \sqrt{\rho_n v_n})\) converges to a weak solution of System 3D-CPEs,
- \((\rho_n, u_n, v_n)\) satisfies the EEI and converges to a weak solution of 3D-CPEs-BC.

M. Ersoy, T. Ngom, M. Sy

**Sketch of the Proof-step 1**

Prove first the stability for the IM’ with IC and BC’,

\[
\begin{aligned}
\partial_t \xi + \text{div}_x (\xi u) + \partial_z (\xi w) &= 0, \\
\partial_t (\xi u) + \text{div}_x (\xi u \otimes u) + \partial_z (\xi u w) + \nabla_x \xi + r\xi|u|u &= 2\tilde{\nu}_1 \text{div}_x (\xi D_x (u)) + \tilde{\nu}_2 \partial_z (\xi \partial_z u), \\
\partial_z \xi &= 0
\end{aligned}
\]

and by the reverse change of variables “transport” the result to the 3D-CPEs.
Sketch of the proof-step 1

Prove first the stability for the IM’ with IC and BC,

\[
\begin{align*}
\partial_t \xi + \text{div}_x (\xi \, u) + \partial_z (\xi \, w) &= 0, \\
\partial_t (\xi \, u) + \text{div}_x (\xi \, u \otimes u) + \partial_z (\xi \, u \, w) + \nabla_x \xi + r \xi |u|u &= 2\tilde{\nu}_1 \text{div}_x (\xi D_x(u)) + \tilde{\nu}_2 \partial_z (\xi \partial_z u), \\
\partial_z \xi &= 0
\end{align*}
\]

and by the reverse change of variables “transport” the result to the 3D-CPEs.

So,

**Definition**

A weak solution of System IM’ on \([0, T] \times \Omega '\), with BC’ and IC, is a collection of functions \((\xi, u, w)\), if \(\xi \in \mathcal{PE}(u, w; z, \xi_0)\) and the following equality holds for all smooth test function \(\varphi\) with compact support such as \(\varphi(T, x, y) = 0\) and \(\varphi_0 = \varphi_{t=0} : \)

\[
A(\xi, u, w; \varphi, dz) = C(\xi, u; \varphi, dz).
\]
Sketch of the proof-step 1

Prove first the stability for the IM’ with IC and BC’,

\[
\begin{align*}
\partial_t \xi + \text{div}_x (\xi \, u) + \partial_z (\xi \, w) &= 0, \\
\partial_t (\xi \, u) + \text{div}_x (\xi \, u \otimes u) + \partial_z (\xi \, u \, w) + \nabla_x \xi + r \xi |u| u &= 2\tilde{\nu}_1 \text{div}_x (\xi D_x (u)) + \tilde{\nu}_2 \partial_z (\xi \partial_z u), \\
\partial_z \xi &= 0
\end{align*}
\]

and by the reverse change of variables “transport” the result to the 3D-CPEs.

So,

**Definition**

A weak solution of System IM’ on \([0, T] \times \Omega’\), with BC’ and IC, is a collection of functions \((\xi, u, w)\), if \(\xi \in \mathcal{PE}(u, w; z, \xi_0)\) and the following equality holds for all smooth test function \(\varphi\) with compact support such as \(\varphi(T, x, y) = 0\) and \(\varphi_0 = \varphi_{t=0}\):

\[
A(\xi, u, w; \varphi, dz) = C(\xi, u; \varphi, dz).
\]

**Difficulty** : show that under suitable sequence of weak solutions, we can pass to the limit in the non-linear term \(\xi \, u \otimes u\) : typically \(\sqrt{\xi} u\) requires strong convergence.
**Theorem**

Let \((\xi_n, u_n, w_n)\) be a sequence of weak solutions of the IM’ with BC’ and IC satisfying an energy and entropy inequality (EEI) such as

\[
\xi_n \geq 0, \quad \xi_0^n \rightarrow \xi_0 \text{ in } L^1(\Omega'), \quad \xi_0^n u_0^n \rightarrow \xi_0 u_0 \text{ in } L^1(\Omega').
\]

Then, up to a subsequence,

- \(\xi_n\) converges strongly in \(C^0(0,T;L^{3/2}(\Omega'))\),
- \(\sqrt{\xi_n} u_n\) converges strongly in \(L^2(0,T;(L^{3/2}(\Omega'))^2)\),
- \(\xi_n u_n\) converges strongly in \(L^1(0,T;(L^1(\Omega'))^2)\) for all \(T > 0\),
- \((\xi_n, \sqrt{\xi_n} u_n, \sqrt{\xi_n} w_n)\) converges to a weak solution of the IM’,
- \((\xi_n, u_n, w_n)\) satisfies the EEI and converges to a weak solution of the IM’ with BC’.

The energy inequality:

\[
\frac{d}{dt} \int_{\Omega'} \left( \frac{\xi u^2}{2} + (\xi \ln \xi - \xi + 1) \right) dx dz + \int_{\Omega'} \xi (2\bar{\nu}_1 |D_x(u)|^2 + \bar{\nu}_2 |\partial_z u|^2) dx dz
+ r \int_{\Omega'} \xi |u|^3 dx dz \leq 0
\]
**Theorem**

Let \((\xi_n, u_n, w_n)\) be a sequence of weak solutions of the IM' with BC' and IC satisfying an energy and entropy inequality (EEI) such as

\[ \xi_n \geq 0, \quad \xi_0^n \to \xi_0 \text{ in } L^1(\Omega'), \quad \xi_0^n u_0^n \to \xi_0 u_0 \text{ in } L^1(\Omega'). \]

Then, up to a subsequence,

- \(\xi_n\) converges strongly in \(C^0(0, T; L^{3/2}(\Omega'))\),
- \(\sqrt{\xi_n} u_n\) converges strongly in \(L^2(0, T; (L^{3/2}(\Omega'))^2)\),
- \(\xi_n u_n\) converges strongly in \(L^1(0, T; (L^1(\Omega'))^2)\) for all \(T > 0\),
- \((\xi_n, \sqrt{\xi_n} u_n, \sqrt{\xi_n} w_n)\) converges to a weak solution of the IM',
- \((\xi_n, u_n, w_n)\) satisfies the EEI and converges to a weak solution of the IM' with BC'.

The entropy inequality:

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega'} (\xi|u + 2\tilde{\nu}_1 \nabla_x \ln \xi|^2 + 2(\xi \log \xi - \xi + 1)) \, dxdz \\
+ \int_{\Omega'} 2\tilde{\nu}_1 \xi |\partial_z w|^2 + 2\tilde{\nu}_1 \xi |Ax(u)|^2 + \tilde{\nu}_2 \xi |\partial_z u|^2 \, dxdz \\
+ \int_{\Omega'} r\xi|u|^3 + 2\tilde{\nu}_1 r|u|u \nabla_x \xi \, dxdz \\
+ \int_{\Omega'} 8\tilde{\nu}_1 |\nabla_x \sqrt{\xi}|^2 \, dxdz = 0.
\]
To prove the stability result on IM’, we proceed as follows:

1. We obtain suitable \textit{a priori} bounds on \((\xi, u, w)\),
   - we get estimates from the energy inequality,
   - we get estimates from the BD-entropy inequality, i.e. a kind of energy with the multiplier \(u + 2\bar{\nu}_1 \nabla x \xi\).

2. We show the compactness of sequences \((\xi_n, u_n, w_n)\) in appropriate space function,
   - we show the convergence of the sequence \(\sqrt{\xi_n}\),
   - we seek bounds of \(\sqrt{\xi_n u_n}\) and \(\sqrt{\xi_n w_n}\),
   - we prove the convergence of \(\xi_n u_n\),
   - we prove the convergence of \(\sqrt{\xi_n u_n}\).

3. We prove that we can pass to the limit in all terms of the IM’,

4. We “transport” this result with the reverse change of variable to the 3D-CPEs. \(\square\)
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1. Prove the existence of weak solutions of the 3D-CPEs
2. Generalize to any anisotropic pair of viscosities
3. Deal with the case of $p = k \rho^\gamma$, $\gamma \neq 1$, $k = cte$ (also the case $k = k(t, x, y)$)
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The PFS Equation are:

\[
\begin{align*}
\partial_t(A) + \partial_x(Q) &= 0 \\
\partial_t(Q) + \partial_x \left( \frac{Q^2}{A} + p(x, A, E) \right) &= -gA \frac{d}{dx} Z(x) + Pr(x, A, E) - G(x, A, E) - gK(x, S) \frac{Q|Q|}{A}
\end{align*}
\]

with \( A = \begin{cases} 
A_{fs} & \text{if FS} \\
A_p & \text{if P}
\end{cases} \)
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FINITE VOLUME (VF) NUMERICAL SCHEME OF ORDER 1

CELL-CENTERED NUMERICAL SCHEME

\[
\frac{\partial}{\partial t} U(t, x) + \frac{\partial}{\partial x} F(x, U) = S(t, x)
\]
Finite Volume (VF) numerical scheme of order 1

Cell-centered numerical scheme

PFS equations under vectorial form:

$$\frac{\partial t}{\partial t} \mathbf{U}(t, x) + \frac{\partial}{\partial x} F(x, \mathbf{U}) = S(t, x)$$

with $\mathbf{U}_i^n$ cte per mesh $\approx \frac{1}{\Delta x} \int_{m_i} \mathbf{U}(t_n, x) \, dx$ and $S(t, x)$ constant per mesh,
Finite Volume (VF) numerical scheme of order 1

Cell-centered numerical scheme

\[ \partial_t U(t, x) + \partial_x F(x, U) = S(t, x) \]

with \( U^n_i \) \( \text{cte per mesh} \) \( \approx \frac{1}{\Delta x} \int_{m_i} U(t_n, x) \, dx \) and \( S(t, x) \) constant per mesh,

Cell-centered numerical scheme:

\[ U_{i}^{n+1} = U_i^n - \frac{\Delta t^n}{\Delta x} (F_{i+1/2} - F_{i-1/2}) + \Delta t^n S(U_i^n) \]

where

\[ \Delta t^n S_i^n \approx \int_{t_n}^{t_n+1} \int_{m_i} S(t, x) \, dx \, dt \]
Finite Volume (VF) numerical scheme of order 1

Upwinded numerical scheme

PFS equations under vectorial form:

\[
\partial_t \mathbf{U}(t, x) + \partial_x \mathbf{F}(x, \mathbf{U}) = \mathbf{S}(t, x)
\]

with \( \mathbf{U}_{i}^{n} \) cte per mesh \( \approx \frac{1}{\Delta x} \int_{m_{i}}^{x_{i}} \mathbf{U}(t_n, x) \, dx \) and \( \mathbf{S}(t, x) \) constant per mesh,

Upwinded numerical scheme:

\[
\mathbf{U}_{i}^{n+1} = \mathbf{U}_{i}^{n} - \frac{\Delta t^n}{\Delta x} \left( \tilde{\mathbf{F}}_{i+1/2} - \tilde{\mathbf{F}}_{i-1/2} \right)
\]
Our goal: define $\mathcal{F}_{i+1/2}$ in order to preserve continuous properties of the PFS-model

Positivity of $A$, conservativity of $A$, discrete equilibrium, discrete entropy inequality
**Choice of the numerical fluxes**

**Our goal**: define $F_{i+1/2}$ in order to preserve continuous properties of the PFS-model

- Positivity of $A$
- Conservativity of $A$
- Discrete equilibrium
- Discrete entropy inequality
OUR GOAL: define $F_{i+1/2}$ in order to preserve continuous properties of the PFS-model.

- Positivity of $A$
- Conservativity of $A$
- Discrete equilibrium
- Discrete entropy inequality
**Our goal**: define $\mathcal{F}_{i+1/2}$ in order to preserve continuous properties of the PFS-model

Positivity of $A$,
conservativity of $A$, discrete equilibrium, discrete entropy inequality

**Our choice**

- VFRoe solver [BEGVF]
- Kinetic solver [BEG10]

---

_C. Bourdarias, M. Ersoy and S. Gerbi._
A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.

_C. Bourdarias, M. Ersoy and S. Gerbi._
A kinetic scheme for transient mixed flows in non uniform closed pipes: a global manner to upwind all the source terms.
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We introduce

\[ \chi(\omega) = \chi(-\omega) \geq 0, \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1, \]
Gibbs Equilibrium or Maxwellian

We introduce

\[ \chi(\omega) = \chi(-\omega) \geq 0, \quad \int \chi(\omega) d\omega = 1, \quad \int \omega^2 \chi(\omega) d\omega = 1, \]

then we define the **Gibbs equilibrium** by

\[ \mathcal{M}(t, x, \xi) = \frac{A(t, x)}{b(t, x)} \chi \left( \frac{\xi - u(t, x)}{b(t, x)} \right) \]

with

\[ b(t, x) = \sqrt{\frac{p(t, x)}{A(t, x)}} \]
**PRINCIPLE**

**MICRO-MACROSCOPIQ RELATIONS**

Since

\[ \chi(\omega) = \chi(-\omega) \geq 0, \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1, \]

and

\[ \mathcal{M}(t, x, \xi) = \frac{A(t, x)}{b(t, x)} \chi \left( \frac{\xi - u(t, x)}{b(t, x)} \right) \]

then

\[ A = \int_{\mathbb{R}} \mathcal{M}(t, x, \xi) d\xi \]

\[ Q = \int_{\mathbb{R}} \xi \mathcal{M}(t, x, \xi) d\xi \]

\[ \frac{Q^2}{A} + \left( \frac{A}{p} b^2 \right) = \int_{\mathbb{R}} \xi^2 \mathcal{M}(t, x, \xi) d\xi \]
(A, Q) is solution of the PFS system if and only if \( M \) satisfy the transport equation:

\[
\frac{\partial}{\partial t} M + \xi \cdot \frac{\partial}{\partial x} M - g\Phi \frac{\partial}{\partial \xi} M = K(t, x, \xi)
\]

where \( K(t, x, \xi) \) is a collision kernel satisfying a.e. \((t, x)\)

\[
\int_{\mathbb{R}} K \, d\xi = 0, \quad \int_{\mathbb{R}} \xi K \, d\xi = 0
\]

and \( \Phi \) are the source terms.

\[\text{B. Perthame.}\]

Kinetic formulation of conservation laws.

Oxford University Press.

The kinetic formulation $(A, Q)$ is solution of the PFS system if and only if $M$ satisfy the transport equation:

$$\partial_t M + \xi \cdot \partial_x M - g \Phi \partial_\xi M = K(t, x, \xi)$$

where $K(t, x, \xi)$ is a collision kernel satisfying a.e. $(t, x)$

$$\int_{\mathbb{R}} K d\xi = 0, \quad \int_{\mathbb{R}} \xi K d\xi = 0$$

and $\Phi$ are the source terms.

General form of the source terms:

$$\Phi = \underbrace{\frac{d}{dx} Z}_{\text{conservative}} + \underbrace{B \cdot \frac{d}{dx} W}_{\text{non conservative}} + \underbrace{K \frac{Q|Q|}{A^2}}_{\text{friction}}$$

with $W = (Z, S, \cos \theta)$
Discretization of source terms

- Recalling that $A, Q$ and $Z, S, \cos \theta$ constant per mesh
- Forgetting the friction: « splitting »...
Recalling that $A, Q$ and $Z, S, \cos \theta$ constant per mesh

forgetting the friction: « splitting »...

Then $\forall (t, x) \in [t_n, t_{n+1}] \times m_i$

$$\Phi(t, x) = 0$$

since

$$\Phi = \frac{d}{dx} Z + B \cdot \frac{d}{dx} W$$
Simplification of the transport equation

- Recalling that $A$, $Q$ and $Z$, $S$, $\cos \theta$ constant per mesh
- Forgetting the friction: «splitting»...

Then $\forall (t, x) \in [t_n, t_{n+1}] \times m_i$

$$\Phi(t, x) = 0$$

Since

$$\Phi = \frac{d}{dx} Z + B \cdot \frac{d}{dx} W$$

$$\implies$$

$$\partial_t M + \xi \cdot \partial_x M = K(t, x, \xi)$$
Simplification of the transport equation

- Recalling that $A, Q$ and $Z, S, \cos \theta$ constant per mesh
- forgetting the friction: «splitting»...

Then $\forall (t, x) \in [t_n, t_{n+1}] \times \mathcal{M}_i$

$$\Phi(t, x) = 0$$

since

$$\Phi = \frac{d}{dx} Z + B \cdot \frac{d}{dx} W$$

$$\implies$$

$$\begin{align*}
\partial_t f + \xi \cdot \partial_x f &= 0 \\
f(t_n, x, \xi) &= M(t_n, x, \xi) \overset{def}{=} \frac{A(t_n, x, \xi)}{b(t_n, x, \xi)} \chi \left( \frac{\xi - u(t_n, x, \xi)}{b(t_n, x, \xi)} \right)
\end{align*}$$

by neglecting the collision kernel
On \([t_n, t_{n+1}] \times m_i\), we have:

\[
\begin{align*}
\partial_t f + \xi \cdot \partial_x f & = 0 \\
f(t_n, x, \xi) & = M^n_i(\xi)
\end{align*}
\]
Discretization of source terms

On $[t_n, t_{n+1}] \times m_i$, we have:

\[
\begin{aligned}
\partial_t f + \xi \cdot \partial_x f &= 0 \\
f(t_n, x, \xi) &= \mathcal{M}_i^n(\xi)
\end{aligned}
\]

i.e.

\[
f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \xi \frac{\Delta t^n}{\Delta x} \left( \mathcal{M}_{i+\frac{1}{2}}^- (\xi) - \mathcal{M}_{i-\frac{1}{2}}^+ (\xi) \right)
\]
Discretization of source terms

On \([t_n, t_{n+1}] \times m_i\), we have:

\[
\begin{align*}
\{ \quad \partial_t f + \xi \cdot \partial_x f &= 0 \\
       f(t_n, x, \xi) &= M_i^n(\xi)
\end{align*}
\]

i.e.

\[
f_{i}^{n+1}(\xi) = M_i^n(\xi) + \xi \frac{\Delta t^n}{\Delta x} \left( M_{i+\frac{1}{2}}^-(\xi) - M_{i-\frac{1}{2}}^+(\xi) \right)
\]

where

\[
U_{i}^{n+1} = \left( \begin{array}{c} A_{i}^{n+1} \\ Q_{i}^{n+1} \end{array} \right) \overset{\text{def}}{=} \int_{\mathbb{R}} \left( \begin{array}{c} 1 \\ \xi \end{array} \right) f_{i}^{n+1}(\xi) \, d\xi
\]
Discretization of source terms

On \([t_n, t_{n+1}] \times m_i\), we have:

\[
\begin{cases}
\frac{\partial_t f + \xi \cdot \partial_x f}{\partial_x f(t_n, x, \xi)} = 0 \\
f(t_n, x, \xi) = M^n_i(\xi)
\end{cases}
\]

i.e.

\[f_{i}^{n+1}(\xi) = M^n_i(\xi) + \xi \frac{\Delta t^n}{\Delta x} \left( M^-_{i+\frac{1}{2}}(\xi) - M^+_{i-\frac{1}{2}}(\xi) \right)\]

or

\[U_{i}^{n+1} = U_{i}^{n} - \frac{\Delta t^n}{\Delta x} \left( \tilde{F}_{i+1/2} - \tilde{F}_{i-1/2} \right)\]

with

\[\tilde{F}_{i \pm \frac{1}{2}} = \int_{\mathbb{R}} \xi \left( \begin{array}{c} 1 \\ \xi \end{array} \right) M_{i \pm \frac{1}{2}}(\xi) d\xi.\]
The microscopic fluxes

Interpretation: potential barrier

\[
\mathcal{M}_{i+1/2}^{-}(\xi) = \begin{cases} \\
\text{positive transmission} \\
\begin{aligned}
\mathcal{M}_i^n(\xi) &+ 1\{\xi<0, \xi^2 - 2g\Delta\Phi^n_{i+1/2} > 0\} \mathcal{M}_{i+1}^n \left(-\sqrt{\xi^2 - 2g\Delta\Phi^n_{i+1/2}}\right)
\end{aligned} \\
\text{negative transmission}
\end{cases}
\]

\[
\Delta\Phi^n_{i+1/2} \text{ barrière de potentiel}
\]
The microscopic fluxes

Interpretation: potential barrier

\[ M_{i+1/2}^- (\xi) = \begin{cases} 1 \{ \xi > 0 \} M_i^n (\xi) & \text{positive transmission} \\ + 1 \{ \xi < 0, \xi^2 - 2g\Delta\Phi_{i+1/2}^n < 0 \} M_{i+1}^n \left( -\sqrt{\xi^2 - 2g\Delta\Phi_{i+1/2}^n} \right) & \text{reflection} \end{cases} \]

\[ M_{i+1}^n \left( -\sqrt{\xi^2 - 2g\Delta\Phi_{i+1/2}^n} \right) \]

negative transmission

\[ \Delta\Phi_{i+1/2}^n \]

barrière de potentiel
The microscopic fluxes

Interpretation: potential barrier

\[ M_{i+1/2}^{-}(\xi) = \begin{cases} \mathbb{1}_{\{\xi > 0\}} M_{i}^{n}(\xi) + \mathbb{1}_{\{\xi < 0, \xi^2 - 2g\Delta\Phi_{i+1/2}^n > 0\}} M_{i+1}^{n}(\xi) & \text{positive transmission} \\ + \mathbb{1}_{\{\xi < 0, \xi^2 - 2g\Delta\Phi_{i+1/2}^n < 0\}} M_{i}^{n}(-\xi) & \text{reflection} \end{cases} \]

\[ M_{i+1/2}^{-}(\xi) + \mathbb{1}_{\{\xi < 0, \xi^2 - 2g\Delta\Phi_{i+1/2}^n > 0\}} M_{i+1}^{n}(\xi) - \sqrt{\xi^2 - 2g\Delta\Phi_{i+1/2}^n} \]

\[ M_{i+1/2}^{+}(\xi) \]

\[ \Delta\Phi_{i+1/2}^n \]

\[ \Delta\Phi_{i+1/2}^n \text{ may be interpreted as a time-dependant slope!} \]
The microscopic fluxes

Interpretation: pente dynamique $\implies$ décéntrement de la friction

\[
\mathcal{M}_{i+1/2}(\xi) = \left\{\begin{array}{ll}
\mathcal{M}_{i}^{n}(\xi) & \text{positive transmission} \\
\mathcal{M}_{i}^{n}(\xi) + \mathcal{M}_{i+1}^{n}(\xi) & \text{reflection} \\
\mathcal{M}_{i+1}^{n}(\xi) & \text{negative transmission}
\end{array}\right.
\]

\[
\mathcal{M}_{i+1/2} = \mathbb{1}_{\{\xi > 0\}} \mathcal{M}_{i}^{n}(\xi) + \mathbb{1}_{\{\xi < 0, \xi^2 - 2g\Delta\Phi_{i+1/2}^{n} < 0\}} \mathcal{M}_{i}^{n}(-\xi)
\]

\[
\mathcal{M}_{i+1/2} = \mathbb{1}_{\{\xi < 0, \xi^2 - 2g\Delta\Phi_{i+1/2}^{n} > 0\}} \mathcal{M}_{i+1}^{n} \left(-\sqrt{\xi^2 - 2g\Delta\Phi_{i+1/2}^{n}}\right)
\]

$\Delta\Phi_{i+1/2}^{n}$ may be interpreted as a time-dependant slope!

... we reintegrate the friction ...
**Upwinding of the source terms**

- **Conservative** $\partial_x W$:
  \[ W_{i+1} - W_i \]

- **Non-conservative** $B\partial_x W$:
  \[ \overline{B}(W_{i+1} - W_i) \]

where

\[ \overline{B} = \int_0^1 B(s, \phi(s, W_i, W_{i+1})) \, ds \]

for the « straight lines paths », i.e.

\[ \phi(s, W_i, W_{i+1}) = sW_{i+1} + (1 - s)W_i, \, s \in [0, 1] \]

---

G. Dal Maso, P. G. Lefloch and F. Murat.  
Definition and weak stability of nonconservative products.  
Numerical properties

With [ABP00]

\[ \chi(\omega) = \frac{1}{2\sqrt{3}} 1_{[-\sqrt{3}, \sqrt{3}]}(\omega) \]

we have:

- Positivity of \( A \) (under a CFL condition),
- Conservativity of \( A \),
- Natural treatment of drying and flooding area.

for example

E. Audusse and M-O. Bristeau and B. Perthame.
Kinetic schemes for Saint-Venant equations with source terms on unstructured grids.
Numerical properties

With [ABP00]

\[ \chi(\omega) = \frac{1}{2\sqrt{3}} 1_{[-\sqrt{3}, \sqrt{3}]}(\omega) \]

we have:

- Positivity of \( A \) (under a CFL condition),
- Conservativity of \( A \),
- Natural treatment of drying and flooding area.

→ non well-balanced scheme with such a \( \chi \)

→ but easy computation of the numerical fluxes

E. Audusse and M-O. Bristeau and B. Perthame.

Kinetic schemes for Saint-Venant equations with source terms on unstructured grids.

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Upwinding of the friction

The « double dam break »

- horizontal pipe: $L = 100\,m$, $R = 1\,m$.
- initial state: $Q = 0\,m^3/s$, $y = 1.8\,m$.
- Symmetric boundary conditions:

\[ \text{Décentré } K_s = 1/100 \]
\[ \text{Centré } K_s = 1/100 \]

\[ \text{Décentré } K_s = 1/10 \]
\[ \text{Centré } K_s = 1/10 \]
Qualitative analysis of convergence

- upstream piezometric head 104 m

- downstream piezometric head:
Convergence
During unsteady flows $t = 100 \text{ s}$

Erreur L2 : Ligne piezométrique au temps $t = 100 \text{ s}$

<table>
<thead>
<tr>
<th>Ordre VFRoe (polyfit)</th>
<th>0.91301</th>
</tr>
</thead>
<tbody>
<tr>
<td>VFRoe (sans polyfit)</td>
<td></td>
</tr>
<tr>
<td>Ordre FKA (polyfit)</td>
<td>0.88039</td>
</tr>
<tr>
<td>FKA (sans polyfit)</td>
<td></td>
</tr>
</tbody>
</table>

$\|y\|_{L^2}$ vs $\ln(\Delta x)$
Convergence

Stationary $t = 500 \text{ s}$

Erreurs L2 : Ligne piezométrique au temps $t = 500 \text{ s}$

Ordre VFRoe (polyfit) = 1.0742
VFRoe (sans polyfit)

Ordre FKA (polyfit) = 1.0371
FKA (sans polyfit)
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1. Study of the convergence with respect to the $\chi$ function
2. Study of the convergence with respect to the paths used to define the non-conservative product
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A Saint-Venant-Exner model

Saint-Venant equations for the hydrodynamic part:

\[
\begin{aligned}
\partial_t h + \text{div}(q) &= 0, \\
\partial_t q + \text{div} \left( \frac{q \otimes q}{h} \right) + \nabla \left( g \frac{h^2}{2} \right) &= -gh \nabla b \\
\end{aligned}
\]

a bedload transport equation for the morphodynamic part:

\[
\partial_t b + \xi \text{div}(q_b(h, q)) = 0
\]
A Saint-Venant-Exner model

Saint-Venant equations for the hydrodynamic part:

\[
\begin{align*}
\partial_t h + \text{div}(q) &= 0, \\
\partial_t q + \text{div}\left(\frac{q \otimes q}{h}\right) + \nabla \left( g \frac{h^2}{2} \right) &= -gh\nabla b
\end{align*}
\]

a bedload transport equation for the morphodynamic part:

\[
\partial_t b + \xi \text{div}(q_b(h, q)) = 0
\]

with

- \( h \) : water height,
- \( q = hu \) : water discharge,
- \( q_b \) : sediment discharge (empirical law: [MPM48], [G81]),
- \( \xi = 1/(1 - \psi) \) : porosity coefficient.

---

E. Meyer-Peter and R. Müller,
Formula for bed-load transport,

A.J. Grass,
Sediment transport by waves and currents,
A Saint-Venant-Exner model

Saint-Venant equations for the hydrodynamic part:

\[
\begin{cases}
    \partial_t h + \text{div}(q) = 0, \\
    \partial_t q + \text{div}\left(\frac{q \otimes q}{h}\right) + \nabla \left(gh^2\right) = -gh\nabla b
\end{cases}
\]

+ 

a bedload transport equation for the morphodynamic part:

\[
\partial_t b + \xi \text{div}(q_b(h, q)) = 0
\]

with

- \( h \): water height,
- \( q = hu \): water discharge,
- \( q_b \): sediment discharge (empirical law: [MPM48], [G81]),
- \( \xi = 1/(1 - \psi) \): porosity coefficient.

Our goal: derive formally this type of equation from a non classical way.
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The morphodynamic part

is governed by the Vlasov equation:

$$\partial_t f + \text{div}_x(vf) + \text{div}_v((F + \vec{g})f) = r \Delta_v f$$

where:

- $f(t, x, v)$ density function of particles
- $\vec{g} = (0, 0, -g)^t$
- $F = \frac{6\pi \mu a}{M} (u - v)$ Stokes drag force with
  - a radius of a particle (assumed constant)
  - $M = \rho_p \frac{4}{3} \pi a^3$ mass of a particle (assumed constant) with $\rho_p$ density of a particle (assumed constant)
- $u$ fluid velocity
- $\mu$ characteristic viscosity of the fluid (assumed constant)
- $r \Delta_v f$ brownian motion of particles where $r$ is the velocity diffusivity
The hydrodynamic part

is governed by the Compressible Navier-Stokes equations

\[\begin{align*}
\partial_t \rho_w + \text{div}(\rho_w u) &= 0, \\
\partial_t (\rho_w u) + \text{div}(\rho_w u \otimes u) &= \text{div}\sigma(\rho_w, u) + \mathcal{F}, \\
p &= p(t, x)
\end{align*}\]  

(1)

where \(\sigma(\rho_w, u)\) is the anisotropic total stress tensor:

\[-pI_3 + 2\Sigma(\rho_w).D(u) + \lambda(\rho_w)\text{div}(u) I_3\]

The matrix \(\Sigma(\rho_w)\) is anisotropic

\[
\begin{pmatrix}
\mu_1(\rho_w) & \mu_1(\rho_w) & \mu_2(\rho_w) \\
\mu_1(\rho_w) & \mu_1(\rho_w) & \mu_2(\rho_w) \\
\mu_3(\rho_w) & \mu_3(\rho_w) & \mu_3(\rho_w)
\end{pmatrix}
\]

with \(\mu_i \neq \mu_j\) for \(i \neq j\) and \(i, j = 1, 2, 3\).
The coupling

As the medium may be heterogeneous, we propose the following inhomogeneous pressure law as:

\[ p(t, x) = k(t, x_1, x_2) \rho(t, x)^2 \quad \text{with} \quad k(t, x_1, x_2) = \frac{gh(t, x_1, x_2)}{4 \rho_f} \]

where \( \rho := \rho_w + \rho_s \) is called mixed density.

We set \( \rho_s \), the macroscopic density of sediments:

\[ \rho_s = \int_{\mathbb{R}^3} f \, dv \]

The last term \( \mathcal{F} \) on the right hand side of CNEs is the effect of the particles motion on the fluid obtained by summing the contribution of all particles:

\[ \mathcal{F} = - \int_{\mathbb{R}^3} F f \, dv + \rho_w \bar{g} = \frac{9 \mu}{2 a^2 \rho_p} \int_{\mathbb{R}^3} (v - u) f \, dv + \rho_w \bar{g}. \]
Boundary conditions

For the hydrodynamic part:

- on the free surface: a normal stress continuity condition
- at the movable bottom: a wall-law condition and continuity of the velocity at the interface $x_3 = b(t, x)$
Boundary conditions

- For the hydrodynamic part:
  - on the free surface: a normal stress continuity condition
  - at the movable bottom: a wall-law condition and continuity of the velocity at the interface $x_3 = b(t, x)$

- For the morphodynamic part:
  - kinetic boundary conditions? (work in progress) replaced by the equation:
    \[
    S = \partial_t b + \sqrt{1 + |\nabla x b|^2} u_{|x_3=b} \cdot n_b
    \]
    and $S - \sqrt{1 + |\nabla x b|^2} u_{|x_3=b} \cdot n_b$ takes into account incoming and outgoing particles.
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Rescaling for both models, “set $\varepsilon = 0$”

Let

- $\sqrt{\theta}$ be the fluctuation of kinetic velocity,
- $\mathcal{U}$ be a characteristic vertical velocity of the fluid,
- $\mathcal{T}$ be a characteristic time,
- $\tau$ be a relaxation time,
- $L$ be a characteristic vertical height,

and

$$B = \frac{\sqrt{\theta}}{\mathcal{U}}, \quad C = \frac{\mathcal{T}}{\tau}, \quad F = \frac{g \mathcal{T}}{\sqrt{\theta}}, \quad E = \frac{2}{9} \left( \frac{a}{L} \right)^2 \frac{\rho_p}{\rho_f} C$$

with the following asymptotic regime:

$$B = O(1), \quad C = \frac{1}{\varepsilon}, \quad F = O(1), \quad E = O(1).$$

---

T. Goudon and P-E. Jabin and A. Vasseur,

Hydrodynamic limit for the Vlasov-Navier-Stokes Equations. I. Light particles regime,

THE “MIXED” MODEL :
Formally, $\varepsilon \to 0$, we obtain:

- Takes the two first moments of the hydrodynamic limit of Vlasov equation

- Rescaled Navier Stokes Equation

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \partial_{x3} (\rho uv) + \nabla_x P &= \\
&= \text{div}_x (\mu_1 (\rho) D_x (u)) + \partial_{x3} \left( \mu_2 (\rho) (\partial_{x3} u + \nabla_x u_3) \right) \\
&\quad + \nabla_x (\lambda(\rho) \text{div}(u)) \\
\partial_t (\rho u_3) + \text{div}_x (\rho u u_3) + \partial_{x3} (\rho u_3^2) + \partial_{x3} P &= \\
&= \text{div}_x \left( \mu_2 (\rho) (\partial_{x3} u + \nabla_x u_3) \right) + \partial_{x3} \left( \mu_3 (\rho) \partial_{x3} u_3 \right) \\
&\quad + \partial_{x3} (\lambda(\rho) \text{div}(u))
\end{align*}
\]

where
\[
P = p + \theta \rho_s
\]

and
\[
\rho = \rho_w + \rho_s.
\]
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Applying an asymptotic analysis to the mixed model: we finally obtain:

\[
\begin{align*}
\partial_t (h\bar{u}) + \text{div}(h\bar{u} \otimes \bar{u}) + \frac{1}{3} \frac{F_r^2}{F^2} \nabla h^2 &= -\frac{h}{F_r^2} \nabla b + \text{div}(hD(\bar{u})) - \begin{pmatrix} \mathcal{K}_1(u) \\ \mathcal{K}_2(u) \end{pmatrix} \\
S &= \partial_t b + \sqrt{1 + |\nabla x b|^2 u_{x_3=b} \cdot n_b}
\end{align*}
\]
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- find appropriate kinematic boundary condition
- generalize this procedure to a real mixed model
- justify such a formal derivation mathematically
Thank you for attention