A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme

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Outline of the talk

1. **Unsteady mixed flows : PFS equations (Pressurized and Free Surface)**
   - Previous works
   - Formal derivation of the free surface and pressurized model
   - A coupling : the PFS-model

2. **Finite Volume discretization**
   - Discretization of the space domain
   - Explicit first order VFRoe scheme
     1. The Case of a non transition point
     2. The Case of a transition point
     3. Update of the cell state
     4. Approximation of the convection matrix

3. **Numerical experiments**

4. **Conclusion and perspectives**

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1 UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)

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- Formal derivation of the free surface and pressurized model
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2 FINITE VOLUME DISCRETIZATION

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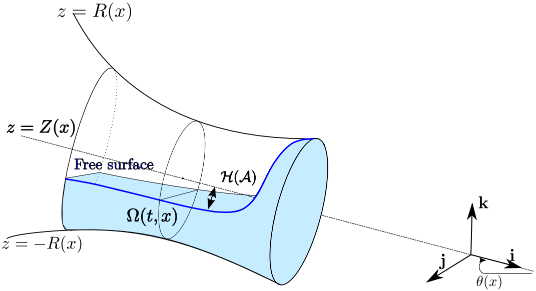
3 NUMERICAL EXPERIMENTS

4 CONCLUSION AND PERSPECTIVES

Unsteady mixed flows in closed water pipes ?

- Free surface area (SL)
  sections are not completely filled and the flow is incompressible. . .



UNSTEADY MIXED FLOWS IN CLOSED WATER PIPES ?

- Free surface area (SL)
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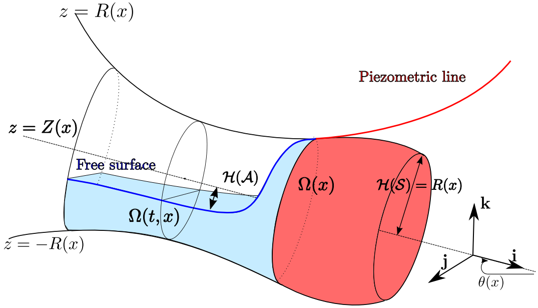
- Pressurized area (CH)
  sections are non completely filled and the flow is compressible. . .



Unsteady mixed flows in closed water pipes ?

- Free surface area (SL)
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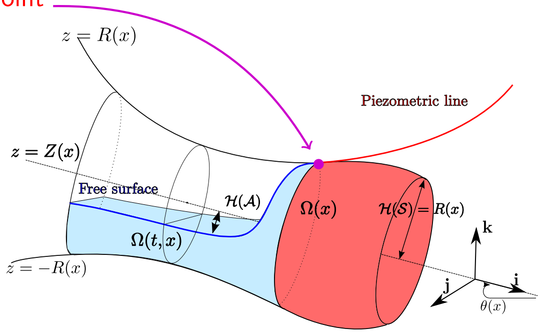
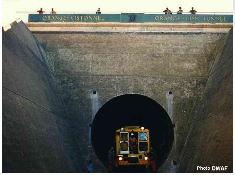
  sections are non completely filled and the flow is compressible...
- Transition point



Examples of pipes



Orange-Fish tunnel

Sewers . . . in Paris

Forced pipe

problems . . . at Minnesota
http://www.sewerhistory.org/grfx/misc/disaster.htm

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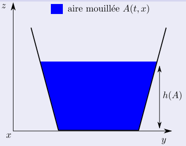
Previous works

For free surface flows :

Generally

Saint-Venant equations :

$$\begin{cases} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(A) \right) = 0 \end{cases}$$



with

$A(t,x)$	:	wet area
$Q(t,x)$	:	discharge
$I_1(A)$	:	hydrostatic pressure
g	:	gravity

Advantage

- Conservative formulation $\longrightarrow$ Easy numerical implementation

Hamam and McCorquodale (82), Trieu Dong (91), Musandji Fuamba (02), Vasconcelos *et al* (06)

Previous works

For pressurized flows :

Generally

Allievi equations :

$$\left\{\begin{array}{l}\partial_t p + \dfrac{c^2}{gS}\partial_x Q = 0,\\ \partial_t Q + gS\partial_x p = 0\end{array}\right.$$

with

$p(t,x)$	:	pressure
$Q(t,x)$	:	discharge
$c(t,x)$	:	sound speed
$S(x)$	:	section

Advantage

- Compressibility of water is taking into account $\Longrightarrow$ Sub-atmospheric flows and over-pressurized flows are well computed

Drawback

- Non conservative formulation $\Longrightarrow$ Cannot be, at least easily, coupled to Saint-Venant equations

Winckler (93), Blommaert (00)

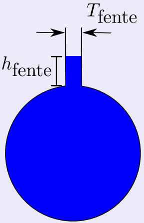
Previous works

For mixed flows :

Generally

Saint-Venant with Preissmann slot artifact :

$$\left\{\begin{array}{l} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + gI_1(A)\right) = 0 \end{array}\right.$$



Advantage

- Only one model for two types of flows.

Drawbacks

- Incompressible Fluid $\Longrightarrow$ Water hammer not well computed
- Pressurized sound speed $\simeq \sqrt{S/T_{\text{fente}}} \Longrightarrow$ adjustment of T_{fente}
- Depression $\Longrightarrow$ seen as a free surface state

Preissmann (61), Cunge *et al.* (65), Baines *et al.* (92), Garcia-Navarro *et al.* (94), Capart *et al.* (97), Tseng (99)

Our goal :

- Use Saint-Venant equations for free surface flows

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- Write a pressurized model
  - which takes into account the compressibility of water
  - which takes into account the depression
  - similar to Saint-Venant equations

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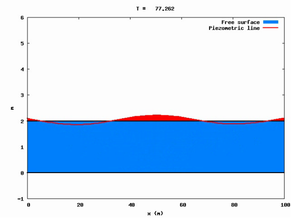
- Use Saint-Venant equations for free surface flows
- Write a pressurized model
  - which takes into account the compressibility of water
  - which takes into account the depression
  - similar to Saint-Venant equations
- Get one model for mixed flows

To be able to simulate, for instance :



C. Bourdarias and S. Gerbi
A finite volume scheme for a model coupling free surface and pressurized flows in pipes.
J. Comp. Appl. Math., 209(1) :109–131, 2007.

Derivation of the free surface model

$3D$ Incompressible Euler equations

$$\begin{aligned} \rho_0 \mathrm{div}(\mathbf{U}) &= 0 \\ \rho_0(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}) + \nabla p &= \rho_0 F \end{aligned}$$

Method :

1. Write Euler equations in curvilinear coordinates.
2. Write equations in non-dimensional form using the small parameter $\epsilon = H/L$ and takes $\epsilon = 0$.
3. Section averaging $\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. Introduce $A_{sl}(t,x)$: wet area, $Q_{sl}(t,x)$ discharge given by :

$$A_{sl}(t,x) = \int_{\Omega(t,x)} dydz, \quad Q_{sl}(t,x) = A_{sl}(t,x)u(t,x)$$

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J.-F. Gerbeau, B. Perthame
Derivation of viscous Saint-Venant System for Laminar Shallow Water ; Numerical Validation.
Discrete and Continuous Dynamical Systems, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

F. Marche
Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects.
European Journal of Mechanic B/Fluid, 26 (2007), 49–63.

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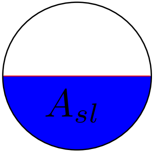

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The free surface model

$$
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\partial_t Q_{sl} + \partial_x \left(\dfrac{Q_{sl}^2}{A_{sl}} + p_{sl}(x, A_{sl}) \right) & = & -g A_{sl} \dfrac{d\,Z}{dx} + Pr_{sl}(x, A_{sl}) - G(x, A_{sl})
\end{array}
\right.
$$

with

$$
\begin{array}{lcll}
p_{sl} & = & g I_1(x, A_{sl}) \cos\theta & : \text{hydrostatic pressure law} \\
Pr_{sl} & = & g I_2(x, A_{sl}) \cos\theta & : \text{pressure source term} \\
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$$p_{sl} = g I_1(x, A_{sl}) \cos\theta \quad : \text{hydrostatic pressure law}$$

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$$K = \frac{1}{K_s^2 R_h(A_{sl})^{4/3}} \quad : \text{Manning-Strickler law}$$

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{U}) = 0$$
$$\partial_t(\rho \mathbf{U}) + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{F}$$

with

$$p = p_a + \frac{\rho - \rho_0}{c^2} \text{ with } c \text{ sound speed}$$

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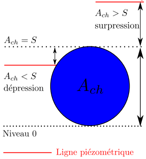

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The pressurized model

$$\begin{cases} \partial_t A_{ch} + \partial_x Q_{ch} & = & 0, \\ \partial_t Q_{ch} + \partial_x \left(\dfrac{Q_{ch}^2}{A_{ch}} + p_{ch}(x, A_{ch}) \right) & = & -g A_{ch} \dfrac{d\, Z}{dx} + Pr_{ch}(x, A_{ch}) - G(x, A_{ch}) \\ & & \\ & & \end{cases}$$

with

$$\begin{array}{lcll} p_{ch} & = & c^2 (A_{ch} - S) & : \text{acoustic type pressure law} \\ Pr_{ch} & = & c^2 \left(\dfrac{A_{ch}}{S} - 1 \right) \dfrac{d\, S}{dx} & : \text{pressure source term} \\ G & = & g A_{ch} \overline{z} \dfrac{d}{dx} \cos\theta & : \text{curvature source term} \end{array}$$

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THE PFS MODEL

MODELS ARE FORMALLY CLOSE ...

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Continuity criterion

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« mixed » condition

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To be coupled

The PFS model

The « mixed » variable

We introduce a state indicator

$$E = \begin{cases} 1 & \text{if the flow is pressurized (CH)}, \\ 0 & \text{if the flow is free surface (SL)} \end{cases}$$

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and the physical section of water **S** by :

$$\mathbf{S} = \mathbf{S}(A_{sl}, E) = \begin{cases} S & \text{if} \quad E = 1, \\ A_{sl} & \text{if} \quad E = 0. \end{cases}$$

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We set

$$\begin{array}{rcll} A & = & \dfrac{\bar{\rho}}{\rho_0}\mathbf{S} = \begin{cases} \mathbf{S}(A_{sl}, 0) = A_{sl} & \text{if} \quad \text{SL} \\ \dfrac{\bar{\rho}}{\rho_0}\mathbf{S}(A_{sl}, 1) = A_{ch} & \text{if} \quad \text{CH} \end{cases} & : \quad \text{the « mixed »variable} \\ Q & = & Au & : \quad \text{the discharge} \end{array}$$

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$$E = \begin{cases} 1 & \text{if the flow is pressurized (CH)}, \\ 0 & \text{if the flow is free surface (SL)} \end{cases}$$

and the physical section of water **S** by :

$$\mathbf{S} = \mathbf{S}(A_{sl}, E) = \begin{cases} S & \text{if} & E = 1, \\ A_{sl} & \text{if} & E = 0. \end{cases}$$

We set

$$\begin{array}{rcll} A & = & \dfrac{\bar{\rho}}{\rho_0}\mathbf{S} = \begin{cases} \mathbf{S}(A_{sl}, 0) = A_{sl} & \text{if} \quad \text{SL} \\ \dfrac{\bar{\rho}}{\rho_0}\mathbf{S}(A_{sl}, 1) = A_{ch} & \text{if} \quad \text{CH} \end{cases} & : \quad \text{the « mixed » variable} \\ Q & = & Au & : \quad \text{the discharge} \end{array}$$

Continuity of **S** at transition point

- Continuity of **S** $\Longrightarrow$ continuity of p at transition point

$\longrightarrow$

$$p(x, A, E) = c^2(A - \mathbf{S}) + gI_1(x, \mathbf{S})\cos\theta$$

- Continuity of $\mathbf{S} \Longrightarrow$ continuity of p at transition point
  $\longrightarrow$
  $$p(x, A, E) = c^2(A - \mathbf{S}) + gI_1(x, \mathbf{S}) \cos\theta$$
- Similar construction for the pressure source term :
  $$Pr(x, A, E) = c^2 \left(\frac{A}{\mathbf{S}} - 1\right) \frac{d\,S}{dx} + gI_2(x, \mathbf{S}) \cos\theta$$

The PFS model

$$\begin{cases} \partial_t(A) + \partial_x(Q) & = 0 \\ \partial_t(Q) + \partial_x\left(\dfrac{Q^2}{A} + p(x,A,E)\right) & = -g\,A\,\dfrac{d}{dx}Z(x) \\ & \quad +Pr(x,A,E) \\ & \quad -G(x,A,E) \\ & \quad -g\,K(x,\mathbf{S})\,\dfrac{Q|Q|}{A} \end{cases}$$

C. Bourdarias, M. Ersoy and S. Gerbi
A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.
Int. J. On Finite Volumes, 6(2) :1–47, 2009.

Mathematical properties

- The **PFS** system is strictly hyperbolic for $A(t,x) > 0$.
- For regular solutions, the mean speed $u = Q/A$ verifies

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + c^2 \ln(A/S) + g\,\mathcal{H}(S) \cos\theta + g\,Z \right) = -g\,K(x, \mathbf{S})\,u\,|u|$$

and for $u = 0$, we have :

$$c^2 \ln(A/\mathbf{S}) + g\,\mathcal{H}(\mathbf{S}) \cos\theta + g\,Z = cte$$

where $\mathcal{H}(\mathbf{S})$ is the physical water height.

- There exists a mathematical entropy

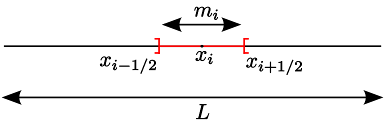
$$E(A,Q,S) = \frac{Q^2}{2A} + c^2 A \ln(A/\mathbf{S}) + c^2 S + g\overline{z}(x,\mathbf{S}) \cos\theta + gAZ$$

which satisfies

$$\partial_t E + \partial_x \left(E\,u + p(x,A,E)\,u \right) = -g\,A\,K(x,\mathbf{S})\,u^2\,|u| \leqslant 0$$

OUTLINE

OUTLINE



Geometric terms and unknowns are piecewise constant approximations on the cell m_i at time t_n :

- Geometric terms
  - Z_i, S_i, $\cos\theta_i$
- unknowns
  - (A_i^n, Q_i^n), $u_i^n = \dfrac{Q_i^n}{A_i^n}$
- Notation : “unknown” vector
  - $\mathbf{W}_i^n = (Z_i, \cos\theta_i, S_i, A_i^n, Q_i^n)^t$

OUTLINE

Adding the equations $\partial_t Z = 0$, $\partial_t \cos\theta = 0$ and $\partial_t S = 0$, the PFS-model under a non conservative form reads :

$$\partial_t \mathbf{W} + D(\mathbf{W})\partial_x \mathbf{W} = TS(\mathbf{W}) \tag{1}$$

where $\mathbf{W} = (Z, \cos\theta, S, A, Q)^t$

$$TS(\mathbf{W}) = \left(0, 0, 0, 0, -g\, K(x, \mathbf{S})\, \frac{Q|Q|}{A}\right)$$

$$D(\mathbf{W}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ gA & gA\mathcal{H}(\mathbf{S}) & \Psi(\mathbf{W}) & c^2(\mathbf{W}) - u^2 & 2u \end{pmatrix}$$

where $\Psi(\mathbf{W}) = gS\partial_S \mathcal{H}(\mathbf{S})\cos\theta - c^2(\mathbf{W})\dfrac{A}{\mathbf{S}}$ and

$$c(\mathbf{W}) = \begin{cases} c & \text{for pressurised flow} \\ \sqrt{g\dfrac{A}{T(A)}\cos\theta} & \text{for free surface flow} \end{cases}$$

Integrating conservative PFS-System over $]x_{i-1/2}, x_{i+\frac{1}{2}}[\times[t_n, t_{n+1}[$, we can write a Finite Volume scheme as follows :

$$\begin{aligned} \mathbf{W}_i^{n+1} &= \mathbf{W}_i^n - \frac{\Delta t^n}{h_i}\Big(\mathbf{F}(\mathbf{W}^*_{i+1/2}(0^-, \mathbf{W}_i^n, \mathbf{W}_{i+1}^n)) - \mathbf{F}(\mathbf{W}^*_{i-1/2}(0^+, \mathbf{W}_{i-1}^n, \mathbf{W}_i^n)) \\ &\quad + TS(\mathbf{W}_i^n) \end{aligned} \tag{1}$$

$\mathbf{W}^*_{i+1/2}(\xi = x/t, \mathbf{W}_i, \mathbf{W}_{i+1})$ is the exact or an approximate solution to the Riemann problem at interface $x_{i+1/2}$.

$W^*(0+, \mathbf{W}_i, \mathbf{W}_{i+1}) = (Z_{i+1}, \cos\theta_{i+1}, S_{i+1}, AP, QP)^t$ and
$W^*(0-, \mathbf{W}_i, \mathbf{W}_{i+1}) = (Z_{i+1}, \cos\theta_{i+1}, S_{i+1}, AM, QM)^t$ depend on two types of interfaces :

- a non transition point : the flow on both sides of the interface is of the same type
- a transition point : the flow changes of type through the interface

1. UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)
   - Previous works
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   - A coupling : the PFS-model

2. FINITE VOLUME DISCRETIZATION
   - Discretization of the space domain
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     1. The Case of a non transition point
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3. NUMERICAL EXPERIMENTS

4. CONCLUSION AND PERSPECTIVES

- approximating the convection matrix $D(\mathbf{W})$ by $\widetilde{D}$,

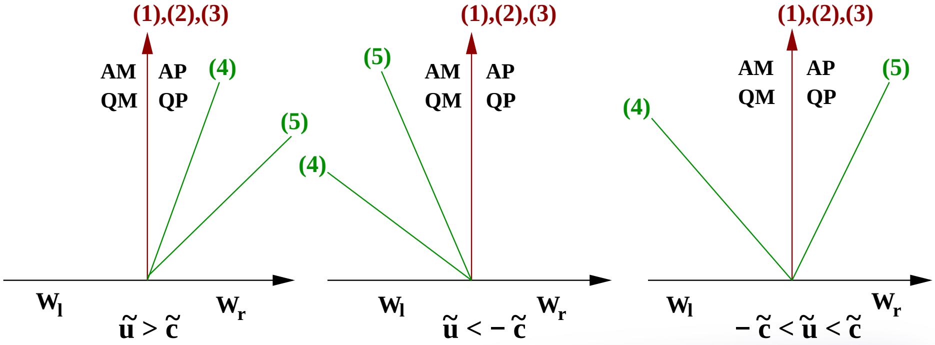
to compute (AM, QM), (AP, QP), we solve the linearized Riemann problem :

$$\left\{\begin{array}{lcl} \partial_t \mathbf{W} + \widetilde{D}\,\partial_x \mathbf{W} & = & 0 \\ \mathbf{W} & = & \left\{\begin{array}{lll} \mathbf{W}_l = (Z_l, \cos\theta_l, S_l, A_l, Q_l)^t & \text{if} & x < 0 \\ \mathbf{W}_r = (Z_r, \cos\theta_r, S_r, A_r, Q_r)^t & \text{if} & x > 0 \end{array}\right. \end{array}\right. \quad (1)$$

with $(\mathbf{W}_l, \mathbf{W}_r) = (\mathbf{W}_i, \mathbf{W}_{i+1})$ and $\widetilde{D} = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}})$ where $\widetilde{\mathbf{W}}$ is some approximate state of the left $\mathbf{W}_l$ and the right $\mathbf{W}_r$ state.

The eigenvalues of $\widetilde{D}$ are $\lambda_i = 0,\ i = 1,2,3,\ \lambda_4 = \widetilde{u} - c(\widetilde{\mathbf{W}}),\ \lambda_5 = \widetilde{u} + c(\widetilde{\mathbf{W}})$

where $c(\mathbf{W}) = \begin{cases} c & \text{for pressurised flow} \\ \sqrt{g\dfrac{A}{T(A)}\cos\theta} & \text{for free surface flow} \end{cases}$



AM, QM, AP, QP ARE GIVEN BY

We obtain, for instance in the sub-critical case (when $-c(\widetilde{\mathbf{W}}) < \widetilde{u} < c(\widetilde{\mathbf{W}})$), we have :

$$AM = A_l + \frac{g\,\widetilde{A}}{2\,c(\widetilde{\mathbf{W}})\,(c(\widetilde{\mathbf{W}}) - \widetilde{u})}\,\psi_l^r + \frac{\widetilde{u} + c(\widetilde{\mathbf{W}})}{2\,c(\widetilde{\mathbf{W}})}\,(A_r - A_l) - \frac{1}{2\,c(\widetilde{\mathbf{W}})}\,(Q_r - Q_l)$$

$$QM = QP = Q_l - \frac{g\,\widetilde{A}}{2\,c(\widetilde{\mathbf{W}})}\,\psi_l^r + \frac{\widetilde{u}^2 - c(\widetilde{\mathbf{W}})^2}{2\,c(\widetilde{\mathbf{W}})}\,(A_r - A_l) - \frac{\widetilde{u} - c(\widetilde{\mathbf{W}})}{2\,c(\widetilde{\mathbf{W}})}\,(Q_r - Q_l)$$

$$AP = AM + \frac{g\,\widetilde{A}}{\widetilde{u}^2 - c(\widetilde{\mathbf{W}})^2}\,\psi_l^r$$

where ψ_l^r is the upwinded source term
$Z_r - Z_l + \mathcal{H}(\widetilde{\mathbf{S}})(\cos\theta_r - \cos\theta_l) + \Psi(\widetilde{\mathbf{W}})(S_r - S_l)$.

OUTLINE

- Assumption the propagation of the interface (pressurized-free surface or free surface-pressurized) has a constant speed w during a time step.
- Consequently the half line $x = w\,t$ is the discontinuity line of $\widetilde{D}(W_l, W_r)$.
- Setting $w = \dfrac{Q^+ - Q^-}{A^+ - A^-}$ with $\mathbf{U}^- = (A^-, Q^-)$ and $\mathbf{U}^+ = (A^+, Q^+)$ the (unknown) states resp. on the left and on the right hand side of the line $x = w\,t$ click.
- Remark Both states $\mathbf{U}_l$ and $\mathbf{U}^-$ (resp. $\mathbf{U}_r$ and $\mathbf{U}^+$) correspond to the same type of flow
- Thus it makes sense to define the averaged matrices in each zone as follows :
  - for $x < w\,t$, we set $\widetilde{D}_l = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_l)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}_l$ and $\mathbf{W}^-$.
  - for $x > w\,t$, we set $\widetilde{D}_r = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_r)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}^+$ and $\mathbf{W}_r$.

Then we formally solve two Riemann problems and use the Rankine-Hugoniot jump conditions through the line $x = w\,t$ which writes :

$$Q^+ - Q^- = w\,(A^+ - A^-) \tag{1}$$

$$F_5(A^+, Q^+) - F_5(A^-, Q^-) = w\,(Q^+ - Q^-) \tag{2}$$

with $F_5(A,Q) = \dfrac{Q^2}{A} + p(x,A)$. According to ($\mathbf{U}^-$, **UM**) and ($\mathbf{U}^+$, **UP**) (unknowns) at the interface $x_{i+1/2}$ and the sign of the speed w, we have to deal with four cases :

- pressure state propagating downstream click, Then we formally solve two Riemann problems and use the Rankine-Hugoniot jump conditions through the line $x = w\,t$ which writes :

$$\begin{aligned} Q^+ - Q^- &= w\,(A^+ - A^-) && (1) \\ F_5(A^+, Q^+) - F_5(A^-, Q^-) &= w\,(Q^+ - Q^-) && (2) \end{aligned}$$

with $F_5(A,Q) = \dfrac{Q^2}{A} + p(x,A)$. According to (**U**$^-$, **UM**) and (**U**$^+$, **UP**) (unknowns) at the interface $x_{i+1/2}$ and the sign of the speed w, we have to deal with four cases :

- pressure state propagating downstream click,
- pressure state propagating upstream,
- free surface state propagating downstream,
- free surface state propagating upstream.

Given n, $\forall i$, A_i^n and E_i^n are known. Then

- if $E_i^n = 0$ then

  if $A_i^{n+1} < S_i$ then $E_i^{n+1} = 0$
  else $E_i^{n+1} = 1$

- if $E_i^n = 1$ then

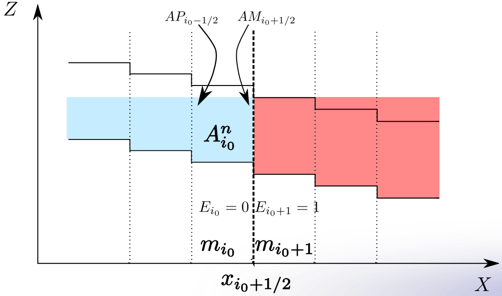
  if $A_i^{n+1} \geqslant S_i$ then $E_i^{n+1} = 1$
  else $E_i^{n+1} = E_{i-1}^n E_{i+1}^n$

The classical approximation $D(\widetilde{\mathbf{W}})$ of the Roe matrix $D_{Roe}(\mathbf{W}_l, \mathbf{W}_r) = \int_0^1 D(\mathbf{W}_r + (1-s)(\mathbf{W}_l - \mathbf{W}_r))\,ds$ defined by $\widetilde{D} = D(\widetilde{\mathbf{W}}) = D\left(\dfrac{\mathbf{W}_l + \mathbf{W}_r}{2}\right)$ preserve the still water steady state only for constant section pipe and $Z = 0$.

Let us start with the consideration : the still water steady state is perfectly maintained : there exists n such that for every i, if $Q_i^n = 0$ and $\forall i$,

$$\begin{aligned}
\text{A1} \ : \ & c^2 \ln\left(\frac{A_{i+1}^n}{S_{i+1}}\right) + g\mathcal{H}(\mathbf{S}_{i+1}^n)\cos\theta + gZ_{i+1} = \\
& c^2 \ln\left(\frac{A_i^n}{S_i}\right) + g\mathcal{H}(\mathbf{S}_i^n)\cos\theta + gZ_i, \\
\text{A2} \ : \ & AM_{i+1/2}^n = AP_{i-1/2}^n, \\
\text{A3} \ : \ & Q_{i+1/2}^n = Q_{i-1/2}^n,
\end{aligned}$$

then, for all $l > n$ the conditions A1, A2 and A3 holds.



$(\widetilde{A}^n_{i-1/2}, \widetilde{A}^n_{i+1/2})$ as the solution of the non-linear system :

$$\left\{\begin{array}{rcl} 0 & = & \Delta A^n_{i+1/2} + \dfrac{g}{2}\left(\dfrac{\widetilde{A}^n_{i+1/2}\psi_i^{i+1}}{\widetilde{c}^2_{i+1/2}} + \dfrac{\widetilde{A}^n_{i-1/2}\psi_{i-1}^{i}}{\widetilde{c}^2_{i-1/2}}\right) \\ 0 & = & \dfrac{g}{2}\left\{\dfrac{\widetilde{A}^n_{i-1/2}\,\psi_{i-1}^{i}}{\widetilde{c}_{i-1/2}} - \dfrac{\widetilde{A}^n_{i+1/2}\,\psi_i^{i+1}}{\widetilde{c}_{i+1/2}}\right\} + \dfrac{\Delta A^n_{i+1/2}}{2}\left(\widetilde{c}_{i-1/2} - \widetilde{c}_{i+1/2}\right) \end{array}\right. \tag{3}$$

the numerical scheme is exactly well-balanced.

For small Δx, we show that

$$\widetilde{A}_{i+1/2}^n \approx \frac{A_i^n + A_{i+1}^n}{2}$$

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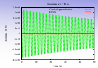
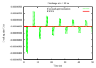
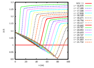
- Well-balanced scheme and the averaged approximation for P



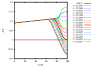
- Well-balanced scheme and the averaged approximation for FS



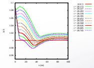
- Depression for a contracting pipe



- Depression for an uniform pipe



- Depression for an expanding pipe



Outline

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Conclusion

- ☑ Conservative and simple formulation (easy implementation even if source terms are complex)
- ☑ Well-balanced numerical scheme
- ☑ Very good agreement for uniform case
- ☑ Compressibility of water for pressurized flows
  - ☑ Water hammer
  - ☑ Depression

Conclusion and perspectives

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Perspectives : cavitation

- condensation
- evaporation

Thank you

for your

attention