A kinetic scheme for transient mixed flows in non uniform closed pipes: a global manner to upwind all the source terms

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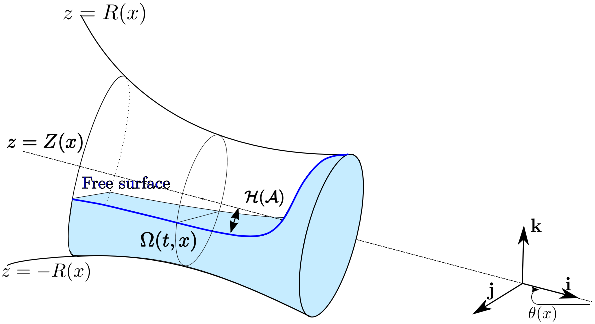
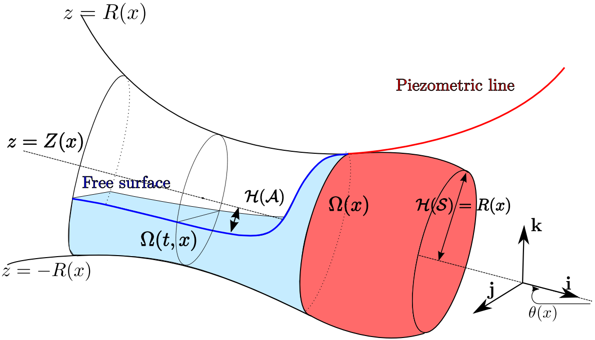
Definition of the mixed flow

- Free surface (FS) area : only a part of the section is filled.



Definition of the mixed flow

- Free surface (FS) area : only a part of the section is filled.
- Pressurized (P) area : the section is completely filled.



PFS-model [BEG09a]

$$
\left\{
\begin{array}{ll}
\partial_t(A) + \partial_x(Q) & = 0 \\
\partial_t(Q) + \partial_x\left(\dfrac{Q^2}{A} + p(x,A,S)\right) & = -g\,A\,\dfrac{d}{dx}Z(x) \\
 & \quad + Pr(x,A,S) \\
 & \quad - G(x,A,S) \\
 & \quad - g\,A\,K(x,S)\,u\,|u|
\end{array}
\right. .
$$

- $A = \dfrac{\rho}{\rho_0} S$: wet equivalent area,
- $Q = A\,u$: discharge,
- S the physical wet area.

The pressure is $p(x,A,S) = c^2\,(A - S) + g\,I_1(x,S)\cos\theta$.

Source terms

- The pressure source term:

$$Pr(x,A,S) = \left(c^2\,(A/S-1)\right)\frac{d}{dx}S + g\,I_2(x,S)\,\cos\theta,$$

- the $z-$coordinate of the center of mass term:

$$G(x,A,S) = g\,A\,\overline{Z}(x,S)\frac{d}{dx}\cos\theta,$$

- the friction term:

$$K(x,S) = \frac{1}{K_s^2\,R_h(S)^{4/3}}.$$

  - $K_s > 0$ is the Strickler coefficient,
  - $R_h(S)$ is the hydraulic radius.

[BEG09a] *C. Bourdarias and M. Ersoy and S. Gerbi.* A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme. Submitted. *Available on arXiv http://arxiv.org/abs/0812.0057, 2009.*

Summarize of notations

- $I_1(x,S) = \displaystyle\int_{-R}^{\mathcal{H}(S)} (\mathcal{H}(S) - z)\sigma \, dz$: the pressure and $I_2(x,S) = \displaystyle\int_{-R}^{\mathcal{H}(S)} (\mathcal{H}(S) - z)\partial_x\sigma \, dz$: the pressure source term with:
  - $R(x)$ the radius,
  - $\sigma(x,z)$ the width of the cross-section,
  - $\mathcal{H}(S)$ the $z-$coordinate of the free surface.
- $c = \dfrac{1}{\sqrt{\beta\rho_0}}$: the sound of speed in the P zones with:
  - ρ_0 the density at atmospheric pressure p_0,
  - β the water compressibility coefficient.
- $\overline{Z}(x,S) = (\mathcal{H}(S) - I_1(x,S)/S)$: the $z-$coordinate of the center of the mass.

Some Properties

- The PFS system is strictly hyperbolic for $A(t,x) > 0$.
- For smooth solutions, the mean velocity $u = Q/A$ satisfies

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + c^2 \ln(A/S) + g\,\mathcal{H}(S)\cos\theta + g\,Z \right) = -g\,K(x,S)\,u\,|u| \; .$$

  and $u = 0$ reads: $c^2 \ln(A/S) + g\,\mathcal{H}(S)\cos\theta + g\,Z = 0$.
- It admits a mathematical entropy

$$E(A,Q,S) = \frac{Q^2}{2A} + c^2 A \ln(A/S) + c^2 S + g\overline{Z}(x,S)\cos\theta + gAZ$$

  which satisfies the entropy inequality

$$\partial_t E + \partial_x \left(E\,u + p(x,A,S)\,u \right) = -g\,A\,K(x,S)\,u^2\,|u| \leqslant 0$$

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The Kinetic Formulation (KF) [P02]

With

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1 \ ,$$

The Kinetic Formulation (KF) [P02]

With

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1 \ ,$$

we define the Gibbs equilibrium

$$\mathcal{M}(t,x,\xi) = \frac{A}{c(A)} \chi\left(\frac{\xi - u(t,x)}{c(A)}\right)$$

with

$$c(A) = \sqrt{g \frac{I_1(x,A)}{A} \cos\theta} \text{ in the FS zones and,}$$

$$c(S) = \sqrt{g \frac{I_1(x,S)}{S} \cos\theta + c^2} \text{ in the P zones.}$$

The Kinetic Formulation (KF) [P02]

We have the macroscopic-microscopic relations:

$$A = \int_{\mathbb{R}} \mathcal{M}(t,x,\xi)\, d\xi$$

$$Q = \int_{\mathbb{R}} \xi \mathcal{M}(t,x,\xi)\, d\xi$$

$$\frac{Q^2}{A} + Ac(A)^2 = \int_{\mathbb{R}} \xi^2 \mathcal{M}(t,x,\xi)\, d\xi$$

The Kinetic Formulation (KF) [P02]

The Kinetic Formulation

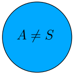
(A, Q) is a strong solution of PFS-System if and only if $\mathcal{M}$ satisfies the kinetic transport equation:

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\Phi(x, A, S)\, \partial_\xi \mathcal{M} = K(t, x, \xi)$$

for some collision term $K(t, x, \xi)$ which satisfies for a.e. (t, x) $\displaystyle\int_{\mathbb{R}} K\, d\xi = 0$, $\displaystyle\int_{\mathbb{R}} \xi\, K d\,\xi = 0$, and Φ which take into account all the source terms.

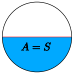
[P02] *B. Perthame.* Kinetic formulation of conservation laws. Oxford University Press. *Oxford Lecture Series in Mathematics and its Applications, Vol 21, 2002.*

The Kinetic Formulation

If 

, Φ reads:

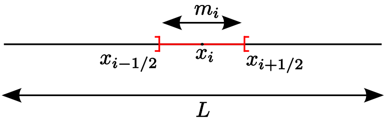
$$\overbrace{\frac{d}{dx}Z - \frac{c^2}{g}\frac{d}{dx}\ln(S)}^{\text{Conservative}} + \overbrace{\overline{Z}(x,S)\frac{d}{dx}\cos\theta}^{\text{Non conservative product}} + \frac{d}{dx}\int_x K(x,S)u|u|\,dx$$

If 

, Φ reads:

$$\overbrace{\frac{d}{dx}Z}^{\text{Conservative}} + \overbrace{\frac{\gamma(x,A)\cos\theta}{A}\frac{d}{dx}\ln(A) + \overline{Z}(x,A)\frac{d}{dx}\cos\theta}^{\text{Non conservative product}} + \frac{d}{dx}\int_x K(x,S)u|u|\,dx$$

Back



Geometric terms and unknowns are piecewise constant approximations on the cell m_i at time t_n:

- Geometric terms
  - S_i, $\cos\theta_i$
- Macroscopic unknowns
  - $\mathbf{W}_i^n = (A_i^n, Q_i^n)$, $u_i^n = \dfrac{Q_i^n}{A_i^n}$
- Microscopic unknown
  - $\mathcal{M}_i^n(\xi) = \dfrac{A_i^n}{c_i^n}\chi\left(\dfrac{\xi - u_i^n}{c_i^n}\right)$

Consequently Φ_i^n is null on m_i.

Indeed, we have:

- $\dfrac{d}{dx}(\mathbb{1}_{m_i} Z) = 0,$
- $\dfrac{d}{dx}(\ln(\mathbb{1}_{m_i} S)) = 0,$
- $\dfrac{d}{dx}(\mathbb{1}_{m_i} \cos\theta) = 0,$
- and we forget the friction term temporarily (friction splitting).

Go

[PS01] *B. Perthame and C. Simeoni*. A kinetic scheme for the Saint-Venant system with a source term. *Calcolo, Vol 38(4) 201–231, 2001*

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Discretisation of the kinetic transport equation

Neglecting the collision term, the transport equation reads on $[t_n, t_{n+1}[\times m_i$:

$$\frac{\partial}{\partial t} f + \xi \cdot \frac{\partial}{\partial x} f = 0$$

with $f(t_n, x, \xi) = \mathcal{M}_i^n(\xi)$ for $x \in m_i$ and thus it is discretised on m_i as:

$$f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \frac{\Delta t^n}{\Delta x} \xi \left(\mathcal{M}^-_{i+\frac{1}{2}}(\xi) - \mathcal{M}^+_{i-\frac{1}{2}}(\xi) \right) ,$$

Although f_i^{n+1} is not a Gibbs equilibrium, we have :

$$\mathbf{W}_i^{n+1} = \begin{pmatrix} A_i^{n+1} \\ Q_i^{n+1} \end{pmatrix} \overset{def}{:=} \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f_i^{n+1}(\xi)\, d\xi$$

$\longrightarrow \mathcal{M}_i^{n+1}$ defined without using the collision kernel : it is a way to perform all collisions at once

Finally the kinetic scheme reads:

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n + \frac{\Delta t^n}{\Delta x} \left(F^-_{i+\frac{1}{2}} - F^+_{i-\frac{1}{2}} \right)$$

with the interface fluxes

$$F^{\pm}_{i+\frac{1}{2}} = \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}^{\pm}_{i+\frac{1}{2}}(\xi)\, d\xi$$

where the microscopic fluxes are defined following e.g. [BEG09b, PS01]:

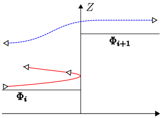
[BEG09b] *C. Bourdarias and M. Ersoy and S. Gerbi.* A kinetic scheme for pressurised flows in non uniform closed water pipes. *Monografias de la Real Academia de Ciencias de Zaragoza, Vol 31 1–20, 2009.*

The microscopic fluxes and physical interpretation

$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\xi>0}\mathcal{M}_i^n(\xi)}^{\text{positive transmission}} + \overbrace{\mathbb{1}_{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}<0}\mathcal{M}_i^n(-\xi)}^{\text{reflection}}$$
$$+ \underbrace{\mathbb{1}_{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}>0}\mathcal{M}_{i+1}^n\left(-\sqrt{\xi^2-2g\Delta\Phi^n_{i+1/2}}\right)}_{\text{negative transmission}}$$

. . .

Reflection & Transmission



$\Delta\Phi^n_{i\pm1/2}$ is the jump condition for a particle with the kinetic speed ξ.

So, $\Delta\Phi^n$ can also seen as a space and time dependent slope.

Geometric source terms

- The friction term.

$$\int_{x_i}^{x_{i+1/2}} \frac{1}{K_s^2} \left\{ \frac{Q|Q|}{A^2 R_h(S)^{4/3}} \right\} dx + \int_{x_{i+1/2}}^{x_{i+1}} \frac{1}{K_s^2} \left\{ \frac{Q|Q|}{A^2 R_h(S)^{4/3}} \right\} dx$$
$$\approx \frac{1}{2\Delta_x K_s^2} \left\{ \frac{Q_{i+1}|Q_{i+1}|}{A_{i+1}^2 R_h(S_{i+1})^{4/3}} + \frac{Q_i|Q_i|}{A_i^2 R_h(S_i)^{4/3}} \right\} := FR_{i+1/2}.$$

- Geometric terms.
  $\partial_x Z$ and $\partial_x \ln(S)$ are easily upwinded

$$Z_{i+1} - Z_i ([\text{PS01}]) \text{ and } \ln\left(\frac{S_{i+1}}{S_i}\right) \quad .$$

Non conservative products

I_2 and $\overline{Z}$ may define non conservative products: using the "straight lines" paths:

$$\phi(s, W_i, W_{i+1}) = sW_{i+1} + (1-s)W_i, \ s \in [0,1]$$

(see e.g. [G01, LT99, DLM95]) with W_i, W_{i+1} the left and right state at the discontinuity $x_{i+1/2}$ permits us to approach any non conservative product $f(x, W)\partial_x W$ as $\overline{f}(W_{i+1} - W_i)$ with the notation

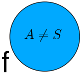
$$\overline{f} = \int_0^1 f(s, \phi(s, W_i, W_{i+1}))\, ds \quad .$$

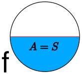
[G01] *L. Gosse*. A well-balanced scheme using non-conservative products designed for hyperbolic systems of conservation laws with source terms. *Math. Models Methods Appl. Sci., Vol 11(2) 339–365, 2001.*

[LT99] *P. G. Lefloch and A.E. Tzavaras*. Representation of weak limits and definition of nonconservative products. *SIAM J. Math. Anal., Vol 30(6) 1309–1342, 1999.*

[DLM95] *G. Dal Maso and P. G. Lefloch and F. Murat*. Definition and weak stability of nonconservative products. *J. Math. Pures Appl. , Vol 74(6) 483–548, 1995.*

$$\Delta\Phi^n_{i+1/2} = \begin{cases} (Z_{i+1} - Z_i) - \dfrac{c^2}{g}\ln\left(\dfrac{S_{i+1}}{S_i}\right) + FR_{i+1/2} & \text{If} \\ (Z_{i+1} - Z_i) - \dfrac{\overline{\gamma(x_{i+1/2}, S)\cos\theta}}{A}\ln\left(\dfrac{A_{i+1}}{A_i}\right) & \\ +\overline{\overline{Z}(x_{i+1/2}, S)}\,(\cos\theta_{i+1} - \cos\theta_i) + FR_{i+1/2} & \text{If} \end{cases}$$





where we make use of the notation $\overline{V} = \dfrac{V_i + V_{i+1}}{2}$ for any quantity V(except $\overline{Z}$).

Properties of the numerical scheme

We choose [ABP00]:

$$\chi(\omega) = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(\omega)$$

We assume a CFL condition. Then

- the kinetic scheme keeps the wetted area A_i^n positive,
- the kinetic scheme preserves the still water steady state,
- Drying and flooding are treated.

[ABP00] *E. Audusse and M-0. Bristeau and B. Perthame*. Kinetic schemes for Saint-Venant equations with source terms on unstructured grids. *INRIA Report RR3989, 2000.*

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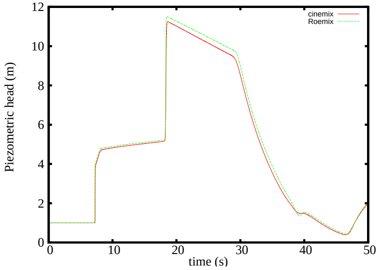
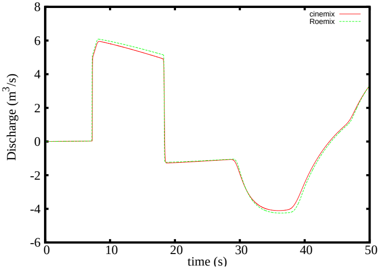
Comparison with the VFROE method [BEG09a]

$R_{upstream} = 0.5$, $R_{downstream} = 0.4$.
Horizontal circular pipe : $L = 1000\ m$.
Inital steady state: $Q = 0\ m^3/s$ and $y = 1\ m$.
Upstream piezometric level is increasing in 5 s at $y = 4\ m$
At downstream : $Q = 0\ m^3/s$



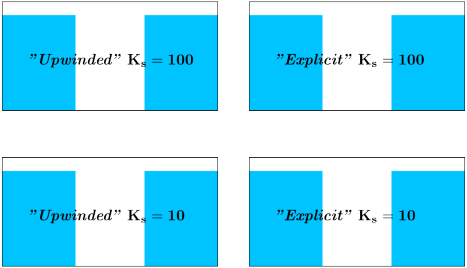
(a) Piezometric level at $x = 500$ m



(b) Discharge at $x = 500$ m

The “double dam break of CB”

Horizontal circular pipe : $L = 100\ m\ R = 1\ m$.
Inital state: $Q = 0\ m^3/s$ and $y = 1.8\ m$.
Upstream and downstream piezometric level is increasing in 30 s at $y = 2.1\ m$



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Conclusion

- Easy implementation of source terms
- Very good agreement for uniform case
- Drying and flooding area are computed

Perspective

- Air entrainment treated as a bilayer fluid flow
- Diphasic approach to take into account air entrapment, evaporation/condensation and cavitation.

Thank you for your attention