Numerical dispersion and Linearized Saint-Venant Equations

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Outline of the talk

Outline

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Motivation

Even if an equation is nondispersive, any discrete model of it will be dispersive[Tref]

L.-N. Trefethen
Group velocity in finite difference schemes.
SIAM Review, 24(1), p. 113–136, 1982.

A simple example : The Saint-Venant equations, approximation of the gravity waves

- The Saint-Venant equations are the equations obtained by vertical averaging of the Navier-Stokes system and are widely used for geophysical fluids, river, lakes, ...
- The most numerical schemes introduce spurious modes ; the most dangerous modes are the stationnary ones :
  - discrete solution may not be unique
  - lead to oscillating solutions
- A fourier analysis is necessary to understand the behavior of discrete modes.

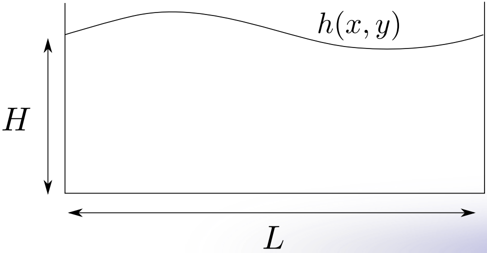
Derivation

it models the shallow water physical configuration where

- the movements are principally horizontal and $\partial_z u = 0$.
- the fluid is assumed incompressible, i.e. $\rho = cte$
- the pressure is hydrostatic, i.e. $\partial_z P = -\rho g$
- the characteristic length L and the height H are such that $H \ll L$

Under these assumptions, a vertical averaging of the Navier-Stokes equations gives :

$$\left\{ \begin{array}{rcl} \partial_t h + H \partial_x u & = & 0 \\ \partial_t u + g \partial_x h & = & 0 \end{array} \right.$$



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Let us consider the preceding system :

$$\left\{ \begin{array}{rcl} \partial_t u + g\partial_x h & = & 0 \\ \partial_t h + H\partial_x u & = & 0 \end{array} \right.$$

As the system is linear, we seek for a solution :

$$\left\{ \begin{array}{rcl} h & = & \tilde{h}\, e^{i(kx+wt)} \\ u & = & \tilde{u}\, e^{i(kx+wt)} \end{array} \right.$$

where

- $\tilde{h}$, $\tilde{u}$: the amplitude
- $kx + wt$: the phase with and where
  - $k = 2\pi/\lambda$: the wave number where λ : the wavelength
  - $w = 2\pi/T$: the frequence where T : the periode

Substituting u and h in the previous equations, we get :

$$\begin{pmatrix} w & gk \\ Hk & w \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{h} \end{pmatrix} = O_{\mathbb{R}^2}.$$

A non identically zero solution is provided when the determinant of this matrix is zero, then we have the following relation :

$$w = \pm\sqrt{gH}k.$$

We deduce

- the phase velocity :

$$v = \frac{w}{k} = \pm\sqrt{gH}$$

- the group velocity :

$$v_g = \frac{\partial w}{\partial k} = \pm\sqrt{gH}$$

As, $v = v_g$, the equations are evidently non dispersive.

Even if an equation is nondispersive ... [Tref]

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. . . any discrete model of it will be dispersive [Tref]

It means that for several numerical schemes, unfortunately, the previous relations are not respected and introduce non physical mode, called spurious mode, which have consequences on the behavior of the solution.

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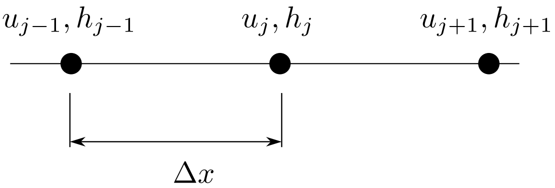
The equations

$$\begin{cases} \partial_t u + g\partial_x h & = & 0 \\ \partial_t h + H\partial_x u & = & 0 \end{cases}$$

are approximated by a cell-centered finite difference scheme where unknowns $u_j(t)$ and $h_j(t)$ are the approximation of $u(t, x_j)$ and $h(t, x_j)$:

$$\begin{cases} \partial_t u_j + g\dfrac{h_{j+1} - h_{j-1}}{2\Delta x} & = & 0 \\ \partial_t h_j + H\dfrac{u_{j+1} - u_{j-1}}{2\Delta x} & = & 0 \end{cases}$$



$$x_j = j\Delta x$$

Substituting u_j and h_j,

$$\left\{\begin{array}{lcl} h_j & = & \tilde{h}\, e^{i(kx_j+wt)} \\ u_j & = & \tilde{u}\, e^{i(kx_j+wt)} \end{array}\right.$$

in the previous discrete equations, we get :

$$\left\{\begin{array}{lcl} iw\tilde{u} + g\tilde{h}\dfrac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} & = & 0 \\ iw\tilde{h} + H\tilde{u}\dfrac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} & = & 0 \end{array}\right.$$

or equivalently

$$\left(\begin{array}{cc} w & g\dfrac{\sin(k\Delta x)}{\Delta x} \\ H\dfrac{\sin(k\Delta x)}{\Delta x} & w \end{array}\right)\left(\begin{array}{c} \tilde{u} \\ \tilde{h} \end{array}\right) = O_{\mathbb{R}^2}.$$

Numerical dispersion

We obvisously get the following frequency

$$w = v\frac{\sin(k\Delta x)}{\Delta x}$$

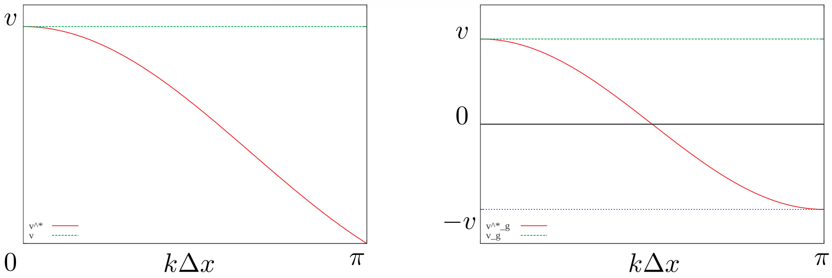
where v is the phase velocity of the continuous model.
We deduce then that :

- the phase velocity for the discrete model is non constant, that is :

$$v^*(k) = w(k)/k = v\frac{\sin(k\Delta x)}{k\Delta x}$$

- the group velocity is

$$v_g^*(k) = \frac{\partial w}{\partial k} = v\cos(k\Delta x)$$



- The phase speed is zero when $k\Delta x = \pi$
- The group speed is negative on the interval $k\Delta x \in [\pi/2, \pi]$

Consequently $\rightarrow$ the energy is propagated in the opposite direction

- For $k\Delta x = \pi$, we have $w = 0$
  - $u_j = \tilde{u}\, e^{ikj\Delta x}$ and
  - $h_j = \tilde{h}\, e^{ikj\Delta x}$.

  Solution is stationnary and do not propagate ! We have :
  - $v^* = 0$
  - $v_g^* = -v$

  and solution oscillates at each nodes.
  Moreover, it is easy to check that this solutions belong to the kernel of the discrete gradient.

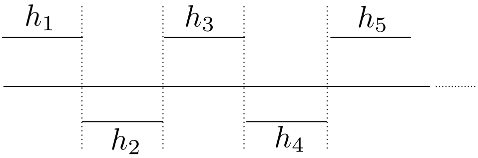
  Consequently $\rightarrow$ it does not allow to get the uniqness of the discrete solution.

  This mode is called spurious mode.
- For $k = 0$, we have again $w = 0$ but, in this case :
  - $v^* = v$
  - $v_g^* = v_g$

  This mode is called hydrostatic mode.

Whenever, spurious mode exists, the solution belong to the kernel of the discrete gradient with $u_j = 0$, $\forall j$, that is : $h_{j+1} = h_{j-1}$, whence we can rewrite as follows :

$$(h_1, h_2, h_3, h_4, \ldots) = h_1 \underbrace{(1, 0, 1, 0, \ldots)}_{d_1} + h_2 \underbrace{(0, 1, 0, 1, \ldots)}_{d_2}$$



Furthemore, we recover the hydrostatic mode for $d_1 + d_2$, i.e. $h_j = h_{j+1}$, $\forall j$.

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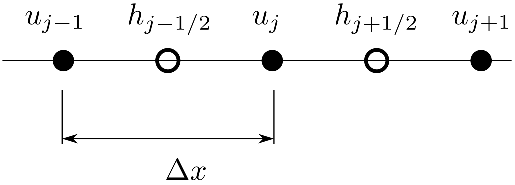
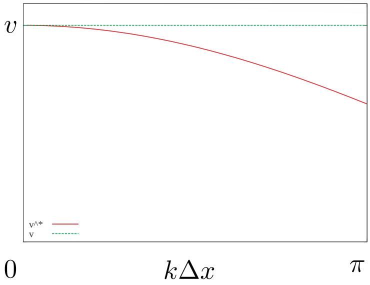
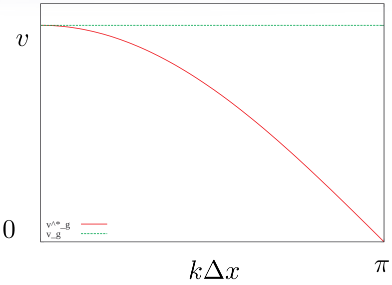
Now, we consider the following discretisation :



$$\begin{cases} \partial_t u_j + g\dfrac{h_{j+1/2} - h_{j-1/2}}{2\Delta x} & = & 0 \\ \partial_t h_{j+1/2} + H\dfrac{u_{j+1} - u_j}{\Delta x} & = & 0 \end{cases}$$





Following the previous computation, we get :

$$\forall kh \in [0,\pi],\ w = v\frac{\sin(\frac{k\Delta x}{2})}{\frac{\Delta x}{2}},\ v^* = v\frac{\sin(\frac{k\Delta x}{2})}{\frac{k\Delta x}{2}},\ c_g^* = v\cos(\frac{k\Delta x}{2}).$$

Contrary to the previous scheme, for $k\Delta x = \pi$, the phase speed is not zero and the energy propagates in the right direction.

Consequently $\rightarrow$ the upwinding of the unknowns on the mesh avoid the apparition of the spurious mode.

In this case, the dimension of the kernel of the discrete gradient is 1, i.e. it contains only the hydrostatic mode.

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We consider again the equations

$$\left\{\begin{array}{rcl} \partial_t u + g\partial_x h & = & 0 \\ \partial_t h + H\partial_x u & = & 0 \end{array}\right.$$

for which we seek solutions under the form :

$$\left\{\begin{array}{rcl} h & = & \tilde{h}(x)\, e^{iwt}, \\ u & = & \tilde{u}(x)\, e^{iwt}, \end{array}\right.$$

that is :

$$\left\{\begin{array}{rcl} iw\tilde{u} + g\partial_x \tilde{h} & = & 0 \\ iw\tilde{h} + H\partial_x \tilde{u} & = & 0 \end{array}\right.$$

Weak formulation

Let Ω be the unidirectionnal domain.
We assume that $\tilde{u} \in V$ and $\tilde{h} \in Q$ where V, Q are $L^2(\Omega)$ or $H^1(\Omega)$.
The weak formulation is : for every smooth test function $\phi \in V$ and $\psi \in Q$, we have

$$\left\{ \begin{array}{rcl} iw \displaystyle\int_\Omega \tilde{u}\phi \, dx + g \int_\Omega \partial_x \tilde{h}\phi \, dx & = & 0 \\ iw \displaystyle\int_\Omega \tilde{h}\psi \, dx + H \int_\Omega \partial_x \tilde{u}\psi \, dx & = & 0 \end{array} \right.$$

Galerkin method

Let τ_h be a discretisation of the domain Ω and Δx the mesh size. For every $K \in \tau_h$, we denote by $P_s(K)$ the space of polynoms of degre s.
For $\tilde{u} \in V_{h|K} = P_1(K)$ and $\tilde{h} \in Q_{h|K} = P_1(K)$, we have :

$$\left\{ \begin{array}{lcl} \dfrac{iw\Delta x}{6}(\tilde{u}_{j-1} + 4\tilde{u}_j + \tilde{u}_{j+1}) + g\dfrac{\tilde{h}_{j+1} - \tilde{h}_{j-1}}{2} & = & 0 \\ \dfrac{iw\Delta x}{6}(\tilde{h}_{j-1} + 4\tilde{h}_j + \tilde{h}_{j+1}) + H\dfrac{\tilde{u}_{j+1} - \tilde{u}_{j-1}}{2} & = & 0 \end{array} \right.$$

we get

$$w = \frac{v}{\Delta x}\left(\frac{3\sin(k\Delta x)}{2 + \cos(k\Delta x)}\right).$$

As in the previous case,

- for $k\Delta x = \pi$, we have $w = 0$ and we are in presence of a spurious and hydrostatic mode.
- "upwinding" unknowns on the mesh will provide the same result as in the "upwinded" finite difference scheme.

Conclusion

- even if the considered model is simple, it easy to see that the numerical dispersion may lead to wrong solution by introducing non physical mode
- Therefore, it is important to understand the meanning of this numerical dispersion, at least , to construct "good" numerical scheme.
- The analysis is done by Fourier analysis

  Consequently $\rightarrow$ only for linear equations with constant coefficients.
  Nevertheless, geometrical optic tools allows to study the behavior of solutions along the bi-characteristic rays

Perspectives

- Developp numerical method to understand, in the context of nonlinear with variable coefficient, the effect of the disrcretisation on the numerical solution such as effect of the numerical dispersion and how the mesh influence it

Further reading

related to shallow water equations

Le Roux D.Y., Pouliot B.
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Int. J. Numer. Methods Fluids, 2008, 57, pp. 951-976.

related to bicharacteristic rays on uniform grids

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Homogenization Limits and Wigner Transforms.
Comm. Pure and Apple. Math., 50 (1997), 323-378.

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and

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Thank you
for your
attention