



A Well Balanced Finite Volume Kinetic (FVK) scheme for unsteady mixed flows in non uniform closed water pipes.

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IMATH, November 24, 2011

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1 UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)

- Previous works
- Formal derivation of the free surface and pressurized model
- A coupling : the PFS-model

2 A FINITE VOLUME FRAMEWORK

- Kinetic Formulation and numerical scheme
- The χ function and well balanced scheme
 1. Classical scheme fails in presence of complex source terms
 2. An alternative toward a Well-Balanced scheme
- Numerical results

3 CONCLUSION AND PERSPECTIVES

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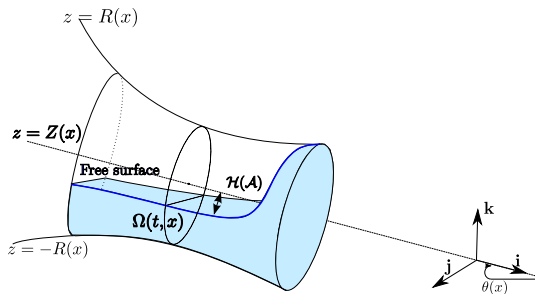
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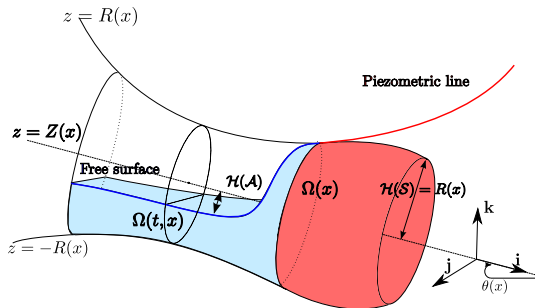
- Free surface area (SL)

sections are not completely filled and the flow is **incompressible**...



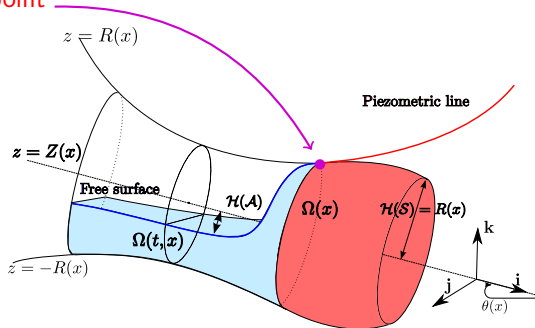
UNSTEADY MIXED FLOWS IN CLOSED WATER PIPES ?

- Free surface area (SL)
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- Pressurized area (CH)
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UNSTEADY MIXED FLOWS IN CLOSED WATER PIPES ?

- Free surface area (SL)
sections are not completely filled and the flow is incompressible. . .
- Pressurized area (CH)
sections are non completely filled and the flow is compressible. . .
- Transition point



EXAMPLES OF PIPES



Orange-Fish tunnel



Sewers ... in Paris



Forced pipe



problems ... at Minnesota

<http://www.sewerhistory.org/grfx/misc/disaster.htm>

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PREVIOUS WORKS

FOR FREE SURFACE FLOWS :

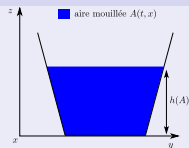
GENERALLY

Saint-Venant equations :

$$\begin{cases} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\frac{Q^2}{A} + g I_1(A) \right) = 0 \end{cases}$$

with

$A(t, x)$	:	wet area
$Q(t, x)$	:	discharge
$I_1(A)$	:	hydrostatic pressure
g	:	gravity



Advantage

- Conservative formulation → Easy numerical implementation



Hamam and McCorquodale (82), Trieu Dong (91), Musandji Fuamba (02), Vasconcelos *et al* (06)

PREVIOUS WORKS

FOR **PRESSURIZED** FLOWS :

GENERALLY

Allievi equations :

$$\begin{cases} \partial_t p + \frac{c^2}{gS} \partial_x Q = 0, \\ \partial_t Q + gS \partial_x p = 0 \end{cases}$$

with

$p(t, x)$	:	pressure
$Q(t, x)$	:	discharge
$c(t, x)$	:	sound speed
$S(x)$	:	section

Advantage

- Compressibility of water is taken into account $\Rightarrow$ Sub-atmospheric flows and over-pressurized flows are well computed

Drawback

- Non conservative formulation $\Rightarrow$ Cannot be, at least easily, coupled to Saint-Venant equations



Winckler (93), Blommaert (00)

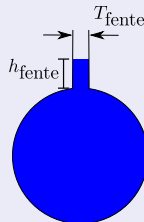
PREVIOUS WORKS

FOR MIXED FLOWS :

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Saint-Venant with Preissmann slot artifact :

$$\begin{cases} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\frac{Q^2}{A} + g I_1(A) \right) = 0 \end{cases}$$



Advantage

- Only one model for two types of flows.

Drawbacks

- Incompressible Fluid $\implies$ Water hammer not well computed
- Pressurized sound speed $\simeq \sqrt{S/T_{\text{fente}}}$ $\implies$ adjustment of T_{fente}
- Depression $\implies$ seen as a free surface state



Preissmann (61), Cunge *et al.* (65), Baines *et al.* (92), Garcia-Navarro *et al.* (94), Capart *et al.* (97), Tseng (99)

OUR GOAL :

- Use Saint-Venant equations for free surface flows

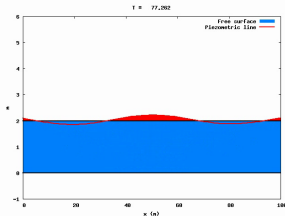
OUR GOAL :

- Use Saint-Venant equations for free surface flows
- Write a pressurized model
 - ▶ which takes into account the compressibility of water
 - ▶ which takes into account the depression
 - ▶ similar to Saint-Venant equations

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- Use Saint-Venant equations for free surface flows
- Write a pressurized model
 - ▶ which takes into account the compressibility of water
 - ▶ which takes into account the depression
 - ▶ similar to Saint-Venant equations
- Get one model for mixed flows

To be able to simulate, for instance :



C. Bourdarias and S. Gerbi

A finite volume scheme for a model coupling free surface and pressurized flows in pipes.

J. Comp. Appl. Math., 209(1) :109–131, 2007.

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DERIVATION OF THE FREE SURFACE MODEL

3D INCOMPRESSIBLE EULER EQUATIONS

$$\begin{aligned}\rho_0 \operatorname{div}(\mathbf{U}) &= 0 \\ \rho_0 (\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) + \nabla p &= \rho_0 F\end{aligned}$$

Method :

- 1 Write Euler equations in curvilinear coordinates.
- 2 Write equations in non-dimensional form using the small parameter $\epsilon = H/L$ and takes $\epsilon = 0$.
- 3 Section averaging $\overline{U^2} \approx \overline{U} \overline{U}$ and $\overline{UV} \approx \overline{U} \overline{V}$.
- 4 Introduce $A_{sl}(t, x)$: wet area, $Q_{sl}(t, x)$ discharge given by :

$$A_{sl}(t, x) = \int_{\Omega(t, x)} dydz, \quad Q_{sl}(t, x) = A_{sl}(t, x)u(t, x)$$

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J.-F. Gerbeau, B. Perthame

Derivation of viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation.
Discrete and Continuous Dynamical Systems, Ser. B, Vol. 1, Num. 1, 89–102, 2001.



F. Marche

Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects.
European Journal of Mechanics B/Fluid, 26 (2007), 49–63.

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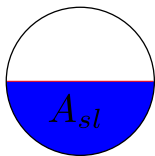
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with

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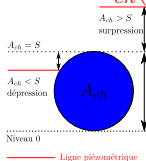
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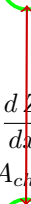
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Continuity criterion

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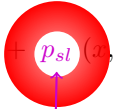
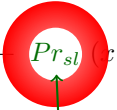
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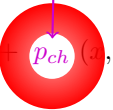
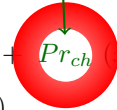
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« mixed » condition

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To be coupled

THE PFS MODEL

THE « MIXED » VARIABLE

We introduce a **state indicator**

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We set

$$\begin{aligned} A &= \frac{\bar{\rho}}{\rho_0} \mathbf{S} = \begin{cases} \mathbf{S}(A_{sl}, 0) = A_{sl} & \text{if SL} \\ \frac{\bar{\rho}}{\rho_0} \mathbf{S}(A_{sl}, 1) = A_{ch} & \text{if CH} \end{cases} : \text{ the « mixed » variable} \\ Q &= Au : \text{ the discharge} \end{aligned}$$

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Continuity of **S** at transition point

THE PFS MODEL

CONSTRUCTION OF THE « MIXED » PRESSURE

- Continuity of $\mathbf{S} \implies$ continuity of p at transition point



$$p(x, A, E) = c^2(A - \mathbf{S}) + gI_1(x, \mathbf{S}) \cos \theta$$

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$$p(x, A, E) = c^2(A - \mathbf{S}) + gI_1(x, \mathbf{S}) \cos \theta$$

- Similar construction for the pressure source term :

$$Pr(x, A, E) = c^2 \left(\frac{A}{\mathbf{S}} - 1 \right) \frac{dS}{dx} + gI_2(x, \mathbf{S}) \cos \theta$$

THE PFS MODEL

$$\left\{ \begin{array}{l} \partial_t(A) + \partial_x(Q) \\ \partial_t(Q) + \partial_x \left(\frac{Q^2}{A} + p(x, A, E) \right) \end{array} \right. = \begin{array}{l} 0 \\ -g A \frac{d}{dx} Z(x) \\ + Pr(x, A, E) \\ -G(x, A, E) \\ -g K(x, \mathbf{s}) \frac{Q|Q|}{A} \end{array}$$



C. Bourdarias, M. Ersoy and S. Gerbi

A model for unsteady mixed flows in non uniform closed water pipes.

SCIENCE CHINA Mathematics, 55(1) :1–26, 2012.

THE PFS MODEL

MATHEMATICAL PROPERTIES

- The **PFS** system is **strictly hyperbolic** for $A(t, x) > 0$.
- For regular solutions, the mean speed $u = Q/A$ verifies

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + c^2 \ln(A/S) + g \mathcal{H}(S) \cos \theta + g Z \right) = -g K(x, \mathbf{S}) u |u|$$

and **for** $u = 0$, we have :

$$c^2 \ln(A/\mathbf{S}) + g \mathcal{H}(\mathbf{S}) \cos \theta + g Z = cte$$

where $\mathcal{H}(\mathbf{S})$ is the physical water height.

- There exists a **mathematical entropy**

$$E(A, Q, S) = \frac{Q^2}{2A} + c^2 A \ln(A/\mathbf{S}) + c^2 S + g \bar{z}(x, \mathbf{S}) \cos \theta + g A Z$$

which satisfies

$$\partial_t E + \partial_x (E u + p(x, A, E) u) = -g A K(x, \mathbf{S}) u^2 |u| \leq 0$$

1 UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)

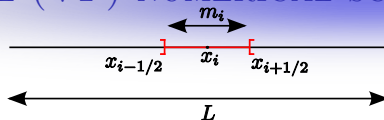
- Previous works
- Formal derivation of the free surface and pressurized model
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2 A FINITE VOLUME FRAMEWORK

- Kinetic Formulation and numerical scheme
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3 CONCLUSION AND PERSPECTIVES

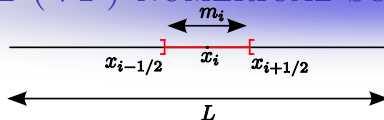
FINITE VOLUME (VF) NUMERICAL SCHEME OF ORDER 1



PFS equations under vectorial form :

$$\partial_t \mathbf{U}(t, x) + \partial_x F(x, \mathbf{U}) = \mathcal{S}(t, x)$$

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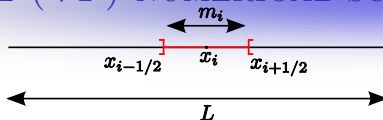


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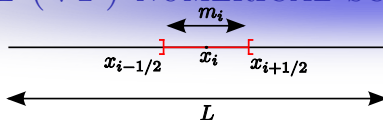
Cell-centered numerical scheme :

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x} (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}) + \Delta t^n \mathcal{S}(\mathbf{U}_i^n)$$

where

$$\Delta t^n \mathcal{S}_i^n \approx \int_{t_n}^{t_{n+1}} \int_{m_i} \mathcal{S}(t, x) dx dt$$

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Upwinded numerical scheme :

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x} \left(\tilde{\mathcal{F}}_{i+1/2} - \tilde{\mathcal{F}}_{i-1/2} \right)$$

$\mathcal{F}$ and $\tilde{\mathcal{F}}$ are consistent.

CHOICE OF THE NUMERICAL FLUXES

Our goal : define $\mathcal{F}_{i+1/2}$ in order to preserve continuous properties of the PFS-model

Positivity of A ,
conservativity of A , discrete equilibrium, discrete entropy inequality

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Our choice

VFRoe solver[BEGVF]

Kinetic solver[BEG10]



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International Journal On Finite Volumes , Vol 6(2) 1–47, 2009.



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A kinetic scheme for transient mixed flows in non uniform closed pipes : a global manner to upwind all the source terms.
J. Sci. Comp., pp 1-16, 10.1007/s10915-010-9456-0, 2011.

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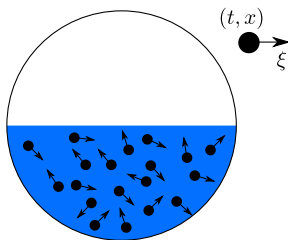
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3 CONCLUSION AND PERSPECTIVES

PHILOSOPHY

As in kinetic theory of gases,

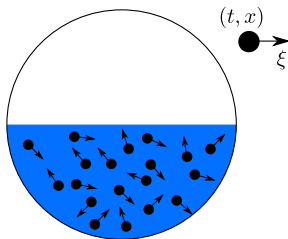
Describe the *macroscopic behavior* from *particle motions*, here, assumed fictitious by introducing $\left\{ \begin{array}{l} \text{a } \chi \text{ density function and} \\ \text{a } \mathcal{M}(t, x, \xi; \chi) \text{ maxwellian function (or a Gibbs equilibrium)} \end{array} \right.$



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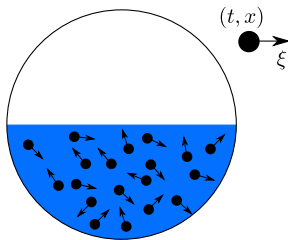


i.e., transform the **nonlinear system** into a **kinetic transport equation** on $\mathcal{M}$.

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i.e., transform the nonlinear system into a kinetic transport equation on $\mathcal{M}$.

Thus, to be able to define the numerical *macroscopic fluxes* from **the** microscopic one.

...Faire d'une pierre deux coups...

PRINCIPLE

DENSITY FUNCTION

We introduce

$$\chi(\omega) = \chi(-\omega) \geq 0, \quad \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \quad \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1,$$

PRINCIPLE

GIBBS EQUILIBRIUM OR MAXWELLIAN

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then we define the **Gibbs equilibrium** by

$$\mathcal{M}(t, x, \xi) = \frac{A(t, x)}{b(t, x)} \chi\left(\frac{\xi - u(t, x)}{b(t, x)}\right)$$

with

$$b(t, x) = \sqrt{\frac{p(t, x)}{A(t, x)}}$$

PRINCIPLE

Since

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then

MICRO-MACROSCOPIC RELATIONS

$$\begin{aligned} A &= \int_{\mathbb{R}} \mathcal{M}(t, x, \xi) d\xi \\ Q &= \int_{\mathbb{R}} \xi \mathcal{M}(t, x, \xi) d\xi \\ \frac{Q^2}{A} + \underbrace{A b^2}_p &= \int_{\mathbb{R}} \xi^2 \mathcal{M}(t, x, \xi) d\xi \end{aligned}$$

PRINCIPLE [P02]

THE KINETIC FORMULATION

(A, Q) is solution of the PFS system if and only if $\mathcal{M}$ satisfy the transport equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g \Phi \partial_\xi \mathcal{M} = \mathcal{K}(t, x, \xi)$$

where $\mathcal{K}(t, x, \xi)$ is a collision kernel satisfying a.e. (t, x)

$$\int_{\mathbb{R}} \mathcal{K} d\xi = 0, \quad \int_{\mathbb{R}} \xi \mathcal{K} d\xi = 0$$

and Φ are the source terms.



B. Perthame.

Kinetic formulation of conservation laws.

Oxford University Press.

Oxford Lecture Series in Mathematics and its Applications, Vol 21, 2002.

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General form of the source terms :

$$\Phi = \overbrace{\frac{d}{dx} Z}^{\text{conservative}} + \overbrace{\mathbf{B} \cdot \frac{d}{dx} \mathbf{W}}^{\text{non conservative}} + \overbrace{K \frac{Q|Q|}{A^2}}^{\text{friction}}$$

with $\mathbf{W} = (Z, S, \cos \theta)$

DISCRETIZATION OF SOURCE TERMS

- Recalling that A, Q and $Z, S, \cos \theta$ constant per mesh
- forgetting the friction : « splitting »...

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Then $\forall (t, x) \in [t_n, t_{n+1}[\times \overset{\circ}{m}_i$

$$\Phi(t, x) = 0$$

since

$$\Phi = \frac{d}{dx} Z + \mathbf{B} \cdot \frac{d}{dx} \mathbf{W}$$

SIMPLIFICATION OF THE TRANSPORT EQUATION

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$\Rightarrow$

$$\begin{cases} \partial_t f + \xi \cdot \partial_x f &= 0 \\ f(t_n, x, \xi) &= \mathcal{M}(t_n, x, \xi) \stackrel{\text{def}}{=} \frac{A(t_n, x, \xi)}{b(t_n, x, \xi)} \chi \left(\frac{\xi - u(t_n, x, \xi)}{b(t_n, x, \xi)} \right) \end{cases}$$

by neglecting the collision kernel.

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On $[t_n, t_{n+1}[\times m_i$, we have :

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$$\mathbf{u}_i^{n+1} = \begin{pmatrix} A_i^{n+1} \\ Q_i^{n+1} \end{pmatrix} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f_i^{n+1}(\xi) d\xi$$

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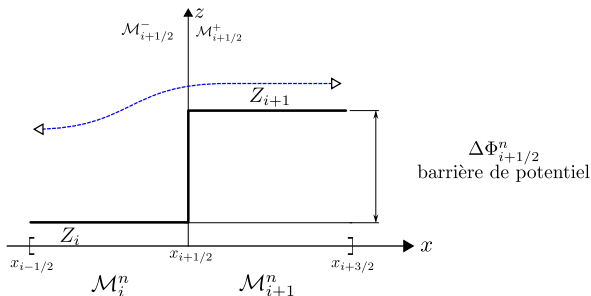
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THE MICROSCOPIC FLUXES

INTERPRETATION : POTENTIAL BAREER

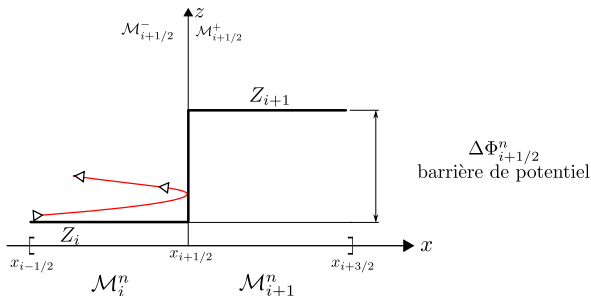
$$\mathcal{M}_{i+1/2}^{-}(\xi) = \overbrace{\mathbb{1}_{\{\xi > 0\}} \mathcal{M}_i^n(\xi)}^{\text{positive transmission}} + \underbrace{\mathbb{1}_{\{\xi < 0, \xi^2 - 2g\Delta\Phi_{i+1/2}^n > 0\}} \mathcal{M}_{i+1}^n\left(-\sqrt{\xi^2 - 2g\Delta\Phi_{i+1/2}^n}\right)}_{\text{negative transmission}}$$



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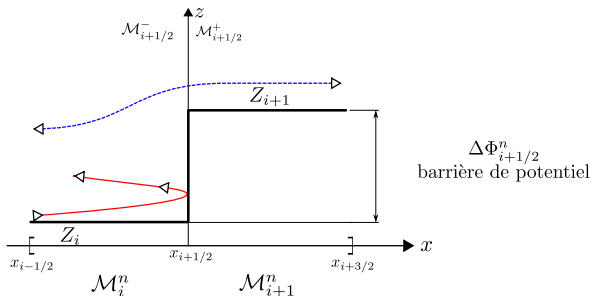
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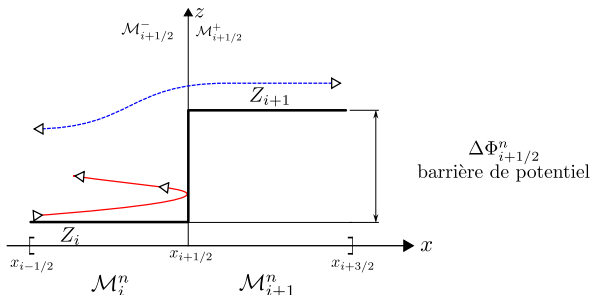


$\Delta\Phi_{i+1/2}^n$ may be interpreted as a time-dependent slope !

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INTERPRETATION : DYNAMIC SLOPE $\implies$ UPWINDING OF THE FRICTION

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... we reintegrate the friction ...

UPWINDING OF THE SOURCE TERMS : $\Delta\Phi_{i+1/2}$

- conservative $\partial_x \mathbf{W}$:

$$\mathbf{W}_{i+1} - \mathbf{W}_i$$

- non-conservative $\mathbf{B}\partial_x \mathbf{W}$:

$$\overline{\mathbf{B}}(\mathbf{W}_{i+1} - \mathbf{W}_i)$$

where

$$\overline{\mathbf{B}} = \int_0^1 \mathbf{B}(s, \phi(s, \mathbf{W}_i, \mathbf{W}_{i+1})) ds$$

for the « straight lines paths », i.e.

$$\phi(s, \mathbf{W}_i, \mathbf{W}_{i+1}) = s\mathbf{W}_{i+1} + (1-s)\mathbf{W}_i, s \in [0, 1]$$



G. Dal Maso, P. G. Lefloch and F. Murat.

Definition and weak stability of nonconservative products.
J. Math. Pures Appl. , Vol 74(6) 483–548, 1995.

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$\chi = ???$ IN PRACTICE $???$

Let us recall that we have to define a χ function such that :

$$\chi(\omega) = \chi(-\omega) \geq 0, \quad \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \quad \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1,$$

and $\mathcal{M} = \frac{A}{b} \chi\left(\frac{\xi - u}{b}\right)$ satisfies the equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g \Phi \partial_\xi \mathcal{M} = 0$$

and

$\chi \longrightarrow$ definition of the macroscopic fluxes.

PROPERTIES RELATED TO χ

We always have

- Conservativity of A holds for every χ .
- Positivity of A holds for every χ but for numerical purpose iff $\text{supp}\chi$ is compact to get a CFL condition.

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while

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- discrete entropy inequalities

strongly depend on the choice of the χ function.

PROPERTIES RELATED TO χ

We always have

- Conservativity of A holds for every χ .
- Positivity of A holds for every χ but for numerical purpose iff $\text{supp}\chi$ is compact to get a CFL condition.

while

- discrete equilibrium,
- discrete entropy inequalities

strongly depend on the choice of the χ function.

In the following, we only focus on discrete equilibrium.

1 UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)

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- A coupling : the PFS-model

2 A FINITE VOLUME FRAMEWORK

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3 CONCLUSION AND PERSPECTIVES

STRATEGY

Even if the **pipe is circular with uniform cross-sections**, for instance for the free surface flows, the following procedure **fails** for complex source terms :

Following [PS01], choose χ such that $\mathcal{M}(t, x, \xi; \chi)$ is the steady state solution at rest, $u = 0$:

$$\xi \cdot \partial_x \mathcal{M} - g \Phi \partial_\xi \mathcal{M} = 0.$$

provides

$$\frac{3 T I_1 - A^2}{2 I_1} w \chi(w) + \left\{ \frac{A^2}{I_1} - w^2 \frac{A^2 - I_1 T}{2 I_1} \right\} \chi'(w) = 0 \text{ where } w = \frac{\xi}{b}.$$



B. Perthame and C. Simeoni

A kinetic scheme for the Saint-Venant system with a source term.
Calcolo, 38(4) :201–231, 2001.

STRATEGY

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Then, this equation is solvable as an ODE iff the coefficients (α, β, γ) are constants.



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For a rectangular pipe with uniform sections, we have $(\alpha, \beta, \gamma) = \left(\frac{T}{2}, 2T, \frac{T}{2} \right)$ with $T = cst$ the base of the pipe.



B. Perthame and C. Simeoni

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Calcolo, 38(4) :201–231, 2001.

IN THESE SETTINGS

With $(\alpha, \beta, \gamma) = \left(\frac{T}{2}, 2T, \frac{T}{2}\right)$ and

THEOREM

we get $\chi(w) = \frac{1}{\pi} \left(1 - \frac{w^2}{4}\right)_+^{1/2}$ and the numerical scheme satisfies the following properties :

- Positivity of A (under a CFL condition),
 - Conservativity of A ,
 - Discrete equilibrium,
 - Discrete entropy inequalities.
-
- This results holds only for conservative terms $\partial_x Z(x)$.
 - A similar result for pressurized flows, unusable in practice (see [PhDErsoy Chap. 2]).



M. Ersoy

Modeling, mathematical and numerical analysis of various compressible or incompressible flows in thin layer [Modélisation, analyse mathématique et numérique de divers écoulements compressibles ou incompressibles en couche mince].

Université de Savoie, Chambéry, September 10, 2010.

IF (α, β, γ) ARE NOT CONSTANTS ...

Then, the equation to solve is :

$$\xi \cdot \partial_x \mathcal{M} - g \Phi \partial_\xi \mathcal{M} = 0.$$

Complicate to solve $\longrightarrow$ find an easy way to maintain, at least, discrete steady states.

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CORRECTION OF THE MACROSCOPIC FLUXES

The steady state is **perfectly maintained** iff

$$\widetilde{\mathcal{F}}_{i+1/2}^{-}(\mathbf{U}_i, \mathbf{U}_{i+1}, \mathbf{Z}_i, \mathbf{Z}_{i+1}) - \widetilde{\mathcal{F}}_{i-1/2}^{+}(\mathbf{U}_{i-1}, \mathbf{U}_i, \mathbf{Z}_{i-1}, \mathbf{Z}_i) = 0$$

with $\mathbf{U} = (A, Q)$, $\mathbf{Z}$ = "source terms"

Notations : $F_{i\pm 1/2}$ the numerical flux of the homogeneous system, $\widetilde{F_{i\pm 1/2}}$ the numerical flux with source term and F the flux of the PFS-model.

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Let us recall that without sources, whenever the numerical flux is consistent, i.e.

$$\forall \mathbf{U} = (A, Q) \in \mathbb{R}^2, F_{i\pm 1/2}(\mathbf{U}, \mathbf{U}) = F(\mathbf{U}),$$

we automatically have, whenever steady states occurs :

$$F_{i+1/2}^{-}(\mathbf{U}_i, \mathbf{U}_{i+1}) - F_{i-1/2}^{+}(\mathbf{U}_{i-1}, \mathbf{U}_i) = 0,$$

i.e.,

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n.$$

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Correction of the numerical flux $\rightarrow$ toward a **well balanced scheme**

Notations : $F_{i\pm 1/2}$ the numerical flux of the homogeneous system, $\widetilde{F_{i\pm 1/2}}$ the numerical flux with source term and F the flux of the PFS-model.

DEFINITION OF THE NEW FLUXES : M-SCHEME

IDEAS : replace

- dynamic quantities $\mathbf{U}_{i-1}$ and $\mathbf{U}_{i+1}$ by stationary profiles $\mathbf{U}_{i-1}^+$ and $\mathbf{U}_{i+1}^-$
- sources terms $\mathbf{Z}_{i-1}$ and $\mathbf{Z}_{i+1}$ by stationary profiles $\mathbf{Z}_{i-1}^+$ and $\mathbf{Z}_{i+1}^-$

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With A_{i+1}^- and A_{i-1}^+ computed from the steady states :

$$\forall i, \begin{cases} D(A_{i+1}^-, Q_{i+1}, \mathbf{Z}_i) &= D(\mathbf{U}_{i+1}, \mathbf{Z}_{i+1}) \\ D(A_{i-1}^+, Q_{i-1}, \mathbf{Z}_i) &= D(\mathbf{U}_{i-1}, \mathbf{Z}_{i-1}) \end{cases}$$

$$\text{where } D(\mathbf{U}, \mathbf{Z}) = \frac{Q^2}{2A} + \begin{cases} g\mathcal{H}(A) \cos \theta + gZ & \text{if } E = 0, \\ c^2 \ln \left(\frac{A}{S} \right) + g\mathcal{H}(S) \cos \theta + gZ & \text{if } E = 1. \end{cases}$$

And $(\mathbf{Z}_{i+1}^-, \mathbf{Z}_{i-1}^+)$ are defined as follows :

$$\mathbf{Z}_{i+1}^- = \begin{cases} \mathbf{Z}_i & \text{if } A_{i+1}^- = A_i \\ \mathbf{Z}_{i+1} & \text{if } A_{i+1}^- \neq A_i \end{cases}$$

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Let us now consider

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t^n}{\Delta x} \left(\mathbf{F}_{i+\frac{1}{2}}^-(\mathbf{U}_i^n, A_{i+1}^-, Q_{i+1}^n, \mathbf{Z}_i, \mathbf{Z}_{i+1}^-) - \mathbf{F}_{i-\frac{1}{2}}^+(A_{i-1}^+, Q_{i-1}^n, \mathbf{U}_i^n, \mathbf{Z}_{i-1}^+, \mathbf{Z}_i) \right)$$

DEFINITION OF THE NEW FLUXES : M-SCHEME

IDEAS : replace

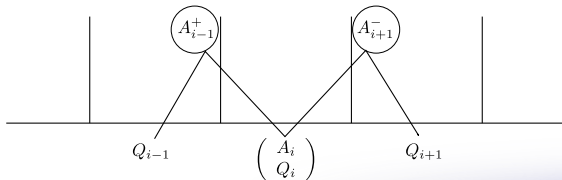
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instead of the previous one :

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t^n}{\Delta x} \left(\mathbf{F}_{i+\frac{1}{2}}^- (\mathbf{U}_i^n, A_{i+1}^n, Q_{i+1}^n, \mathbf{Z}_i, \mathbf{Z}_{i+1}^-) - \mathbf{F}_{i-\frac{1}{2}}^+ (A_{i-1}^n, Q_{i-1}^n, \mathbf{U}_i^n, \mathbf{Z}_{i-1}^-, \mathbf{Z}_i) \right)$$



DEFINITION OF THE NEW FLUXES : M-SCHEME

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Then,

THEOREM

the numerical scheme is well-balanced.

PROOF

- the numerical flux is, by construction, consistent.

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- Let us assume that there exists n such that for every i :

$$Q_i^n = Q_0, \quad D(\mathbf{U}_i^n, \mathbf{Z}_i) = h_0.$$

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Then,

$$D(A_{i+1}^-, Q_{i+1}, \mathbf{Z}_i) = D(\mathbf{U}_{i+1}, \mathbf{Z}_{i+1}) = h_0, \quad \forall i$$

and especially, we have :

$$D(A_{i+1}^-, Q_{i+1}, \mathbf{Z}_i) = D(\mathbf{U}_i, \mathbf{Z}_i).$$

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$$D(A_{i+1}^-, Q_{i+1}, \mathbf{Z}_i) = D(\mathbf{U}_i, \mathbf{Z}_i).$$

The application $A \rightarrow D(A, Q, Z)$ being injective, provides $A_{i+1}^- = A_i$ and thus $\mathbf{Z}_{i+1}^- = \mathbf{Z}_i$ by construction. Similarly, we get $A_{i-1}^+ = A_i$ and $\mathbf{Z}_{i-1}^+ = \mathbf{Z}_i$.

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$$\mathbf{F}_{i+\frac{1}{2}}^-(\mathbf{U}_i^n, \mathbf{U}_{i+1}^-, \mathbf{Z}_i, \mathbf{Z}_{i+1}^-) - \mathbf{F}_{i-\frac{1}{2}}^+(\mathbf{U}_{i-1}^+, \mathbf{U}_i^n, \mathbf{Z}_{i-1}^+, \mathbf{Z}_i) = 0,$$

we get $\forall l \geq n, \quad Q_i^{l+1} = Q_i^l := Q_0$.



NUMERICAL PROPERTIES

For instance, with the simplest χ function [ABP00],

$$\chi(\omega) = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3}, \sqrt{3}]}(\omega)$$

the following properties holds :

- Positivity of A (under a CFL condition),
- Conservativity of A ,
- Discrete equilibrium and,
- Natural treatment of drying and flooding area.

for example

and analytical expression of the numerical macroscopic fluxes.



E. Audusse and M-O. Bristeau and B. Perthame.

Kinetic schemes for Saint-Venant equations with source terms on unstructured grids.
INRIA Report RR3989, 2000.

1 UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)

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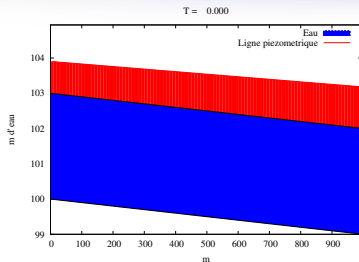
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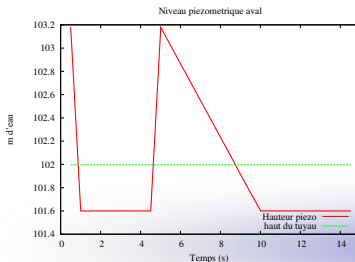
3 CONCLUSION AND PERSPECTIVES

QUALITATIVE ANALYSIS OF CONVERGENCE

AND COMPARISON WITH THE WELL-BALANCED VFROE SCHEME



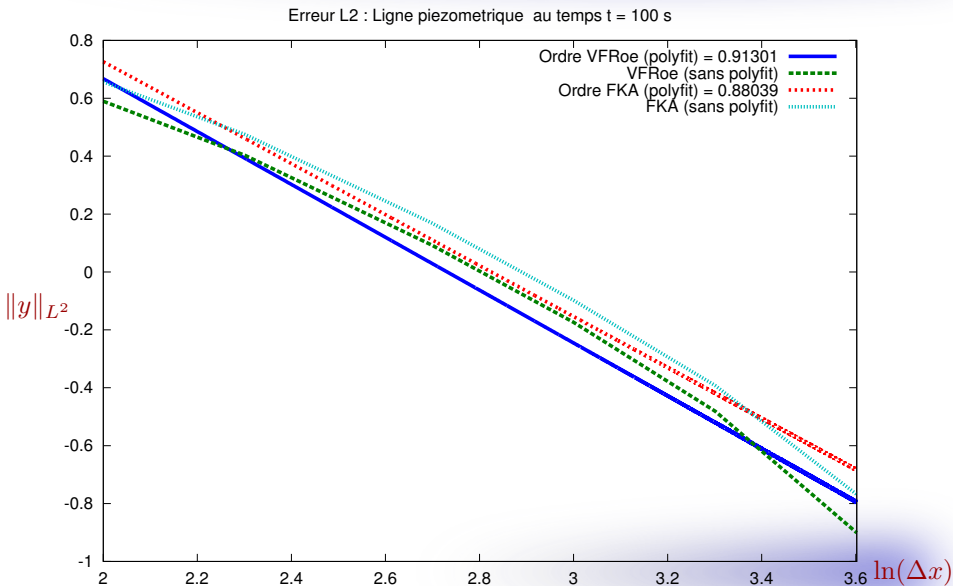
- upstream piezometric head 104 m



- downstream piezometric head :

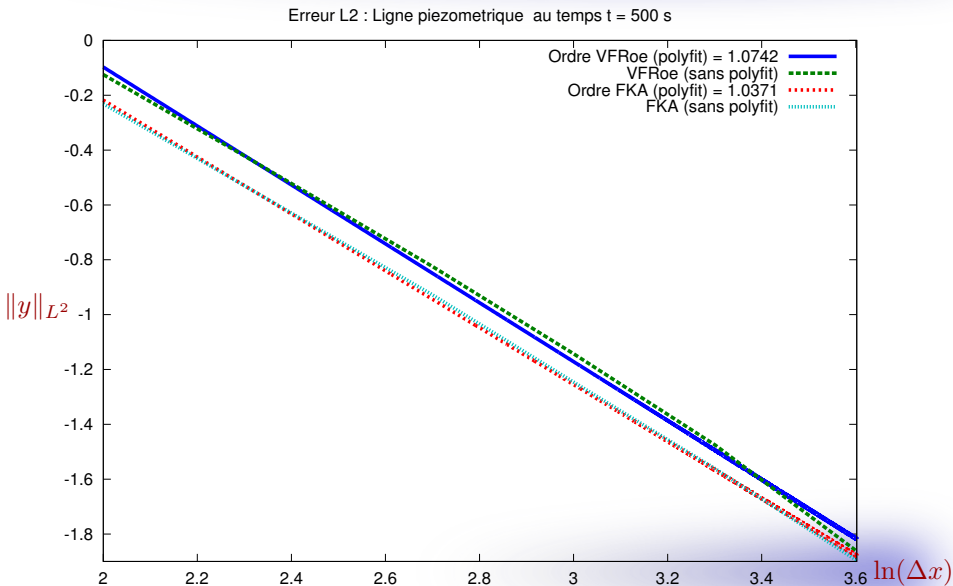
CONVERGENCE

During unsteady flows $t = 100$ s



CONVERGENCE

Stationary $t = 500$ s



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CONCLUSION



Conservative and simple formulation :

→ easy implementation even if source terms are complex



The most of the properties of the continuous model are maintained at discrete level :

→ positivity of the water area

→ conservativity of the water area

→ discrete equilibrium maintained

CONCLUSION AND PERSPECTIVES



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The most of the properties of the continuous model are maintained at discrete level :

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→ discrete equilibrium maintained

What about discrete entropy inequalities ?

→ same difficulties as for discrete balance (see [PhDErsoy] Chap. 2 for further details)

- **Les écoulements mixtes en conduites fermées à géométrie variable.**

Song 76, Perthame-Simeoni 01, Audusse-Bouchut-Bristeau-Klein-Perthame 04.

- ▶ **Modélisation** (Prise en compte de changements d'état).
- ▶ **Anal. math.** (couplage, problème de Riemann).
- ▶ **Anal. num.** (suivi d'interface, sch. vol. finis-cinét., bien-équilibré, entropique).

- **La dynamique de l'atmosphère.**

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- **Lois de conservation scalaire stationnaire et contrôle.**

Jameson 82, Bressan-Marson 95, Bouchut-James 98, Godlewski-Raviart 99.

- ▶ **Anal. math.** (temps long, solution stationnaire entropique)
- ▶ **Anal. num.** (Alternating Descent Method (ADM)).

A dynamic background image showing a large splash of water with many droplets and ripples, creating a sense of movement and freshness. The water is a clear, light blue color.

Thank you

Thank you

for your

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attention

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