Shallow water equations : Modeling, numerics and applications.

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1. Introduction
    - Applications
    - Derivation of SWE
    - Properties of SWE and numerical illustration

2. Numerical approximation of SWE
    - Finite volume scheme for Homogenous SWE
    - Finite volume scheme for non Homogenous SWE
    - Kinetic scheme

3. some physical applications derived from SWE
    - A sediment transport model
    - An unsteady mixed flows in closed water pipes model (PFS model)

4. Concluding remarks

Outline

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The shallow water equations (SWE), first introduced [SV] by Adhémar Jean Claude Barré de Saint-Venant are also called **Saint Venant equations**.

Figure: Adhémar Jean Claude Barré de Saint-Venant (1797-1886)

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Figure: Adhémar Jean Claude Barré de Saint-Venant (1797-1886)

The SWE are a set of *hyperbolic partial differential equations* (or parabolic if viscous shear is considered) obtained from the Euler equations (Navier-Stokes equations) under the following assumptions :

- large-scale assumption ($\varepsilon = \dfrac{H}{L}$ where H being a caracteristic height and L a caracteristic length of the domain)
- and hydrostatic approximation $\partial_z p = -g$.

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Physical applications

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Open channel flows

Water pipe breaking

Based on the same principle, one can found a compressible variant of Saint-Venant equation (as in [BEG11] for unsteady mixed flows in closed water pipes)

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Dam break

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OUTLINE

Equations

The conservative ($S_h = S_{hu} = 0$) 1D SWE are :

$$\left\{\begin{array}{lcl} \partial_t h + \partial_x(hu) & = & S_h \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) & = & S_{hu} \end{array}\right.$$

where

$h(t,x)$	:	density
$u(t,x)$	:	velocity of the water column
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For instance,

- S_h can be a raining term
- S_{hu} can be $-gh\partial_x Z + S_f$ where Z stands for the bathymetry (eventually depends on time) and S_f a friction term

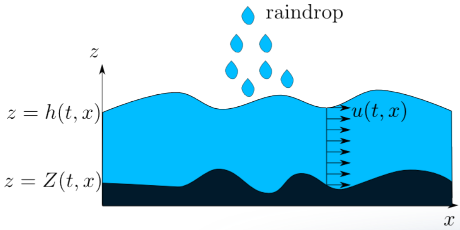
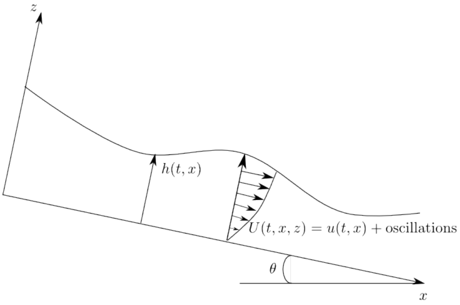


FIGURE: configuration

Almost based on the following assumptions :

- Almost flat bottom



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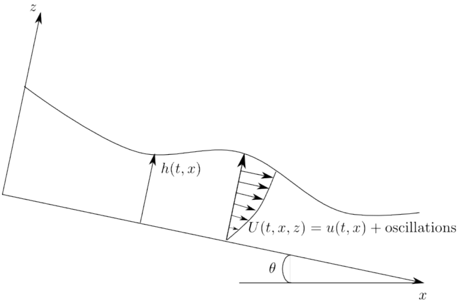

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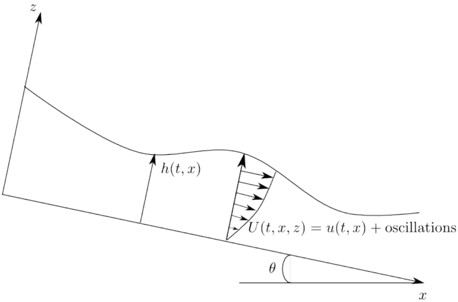

FIGURE: Almost flat bottom configuration

- Constant velocity in z
- Rectangular channel

A mathematical rigorous derivation :

- [GP01] a 1D viscous (non viscous) Saint-Venant system is obtained
- [M07] a 2D viscous (non viscous) Saint-Venant system is obtained

$$\left\{\begin{array}{lcl} \partial_t h + \mathsf{div}_{x,y}(hu) & = & 0 \\ \partial_t(hu) + \mathsf{div}_{x,y}(hu \otimes u + gh^2/2I_2) & = & -gh\nabla_{x,y}Z - 2\Omega \times hu - \kappa(h,u)u \end{array}\right.$$

  where $U(t,x,y,z) = u(t,x) + O(\varepsilon)$
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under the assumptions :

- Shallow water assumption $\rightarrow$ hydrostatic hypothesis
- Scaled viscosity and friction $\rightarrow$ almost constant velocity
- Almost flat bottom and free surface

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SWE

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- Conservation law
- Hyperbolic system (wave propagation, weak solution)
- Entropy inequality
- Positivity of water depth (invariant domain, dry zones, flooding zones)
- Non-trivial steady states

- The conservative SWE

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- The quasi-linear form of the SWE

$$\partial_t W + A(W)\partial_x W = S$$

where $W = \begin{pmatrix} h \\ hu \end{pmatrix}, A(W) = \begin{pmatrix} 0 & 1 \\ u^2 - gh & 2u \end{pmatrix}, S = \begin{pmatrix} 0 \\ -gh\partial_x Z \end{pmatrix}$

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- SW system is strictly hyperbolic on the set $\{(t,x); h(t,x) > 0\}$.

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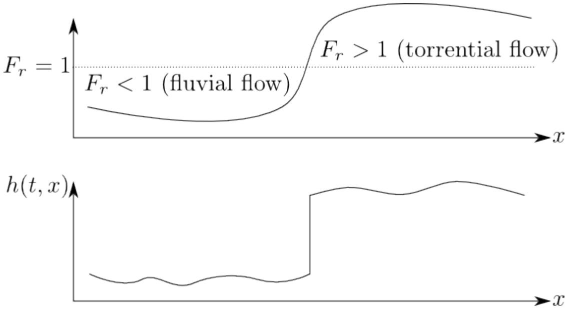
Moreover, one has a hydraulic bore (discontinuous solution) whenever



a hydraulic bore (jump)

kitchen sink

It is well-known that even if the initial data are smooth, the solutions of

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Uniqueness is recovered (weak physical solution) by completing the SW system with an entropy inequality of the form :

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$$s(w) = h\frac{u^2}{2} + gh\left(\frac{h}{2} + Z\right) \text{ and } \psi(w) = u(s(w) + g\frac{h^2}{2}) \ .$$

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Relevant mathematical (physical) solutions, in addition to

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- entropic scheme ,steady states preservation scheme are open problems
- positivity of the water height $\rightarrow$ technical difficulties.

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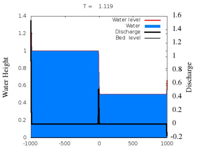
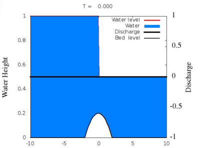
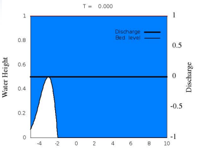


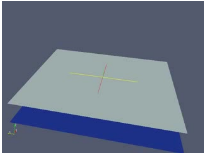
Dambreak on horizontal plane



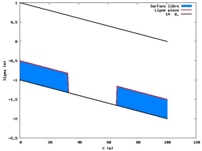
Dambreak over a non constant bottom and transcritical steady solution with shock



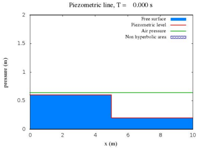
1D moving bottom



2D moving bottom



Dambreak on inclinated plane and drying and fooding phenomena (closed pipe)



A air entrainment model (closed pipe)

Outline

OUTLINE

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- well-adapted for discontinuous solution
- easy implementation on structured or non-structured mesh

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Principle :

- notations :

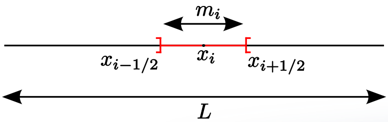

Figure: Mesh

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Principle :

- notations :
  - $u_i^n = \int_{m_i} u(t_n, x)\, dx$: finite volume approximation

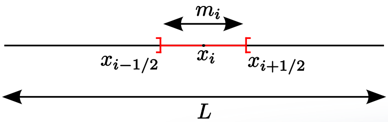

Figure: Mesh

Finite Volume method is designed for conservation laws

$$\partial_t u + \partial_x f(u) = 0$$

- well-adapted for discontinuous solution
- easy implementation on structured or non-structured mesh

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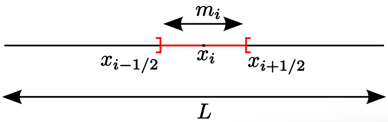

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- integration over the strip $[t_n, t_{n+1}] \times m_i$

$$u_i^{n+1} = u_i^n - \frac{\Delta t_n}{\Delta x} \left(F_{i+1/2} - F_{i-1/2}\right)$$

where $t_{n+1} = t_n + \Delta t_n$ and

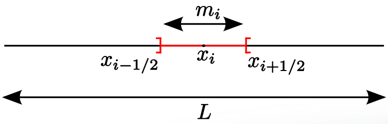

Figure: Mesh

- Explicit scheme are restricted by the time step (CFL, Courant, Friederich and Levy condition (1928))

$$\Delta t_n \leqslant \frac{\Delta x}{\max_i |\lambda_i^n|}$$

- Why ?

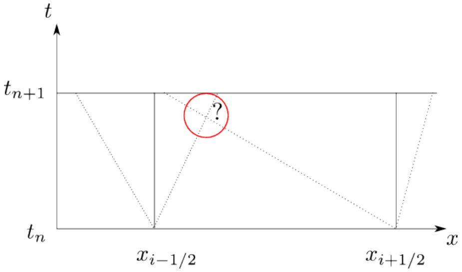

Figure: CFL condition

Introduction of ghost cells to deal with

- the most usual case
  - wall conditions ($u(x_{ghost}, t) = 0$)
  - free boundaries conditions ($u(x_{ghost}, t) = u(0, t)$ and $h(x_{ghost}, t) = h(0, t)$)

with possibly varying in time for upstream and downstream conditions.

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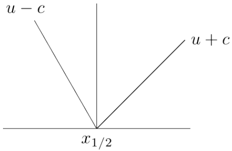

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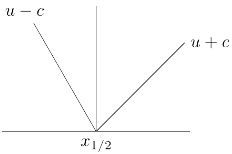
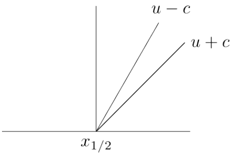
  - upstream and/or downstream torrential in flow : two prescribed value



fluvial flow



upstream torrential in

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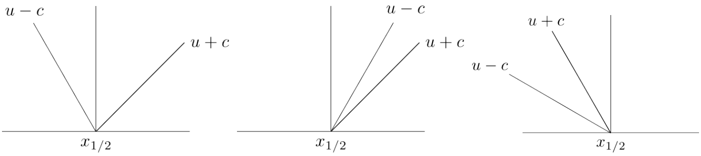
  - upstream and/or downstream torrential in (resp. out) flow : two (resp. free boundary conditions) prescribed value



fluvial flow | upstream torrential in | upstream torrential out

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The simplest choice :

$$F(u,v) = \frac{f(u) + f(v)}{2}$$

lead to unstable numerical scheme (even for the transport equation) !
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Solvers are

- either based on the exact solution of the Riemann problem, namely, the hyperbolic system with the following initial data

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  - ...
- VFRoe (based on the exact solution of the linearized Riemann problem)

- **Principle**
  - solve locally, for each interface $x = x_{i+1/2}$, the Riemann problem with the data

$$u^n(x) = \begin{cases} u_i^n & \text{if} \quad x \leqslant x_{i+1/2} \\ u_{i+1}^n & \text{if} \quad x > x_{i+1/2} \end{cases}$$

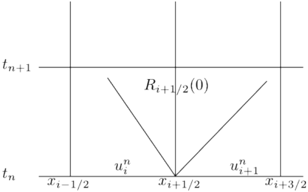

FIGURE: local Riemann problem

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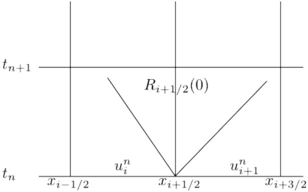

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  - Call $R_{i+1/2}(x/t; u_i^n, u_{i+1}^n)$ the exact solution
  - Define the numerical flux $F_{i+1/2}(u_i^n, u_{i+1}^n) := f(R_{i+1/2}(0))$

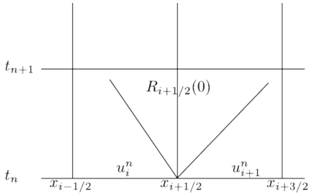

Figure: local Riemann problem

- Rusanov (also known as Lax–Friedrichs scheme)

$$F(u,v) = \frac{f(u)+f(v)}{2} - c\frac{v-u}{2}$$

for some parameter $c > 0$

- HLL flux

$$F(u,v) = \begin{cases} f(u) & \text{if} \quad 0 < c_1 \\ \dfrac{c_2 f(u) - c_1 f(v)}{c_2 - c_1} + \dfrac{c_1 c_2}{c_2 - c_1}(v-u) & \text{if} \quad c_1 < 0 < c_2 \\ f(v) & \text{if} \quad c_2 < 0 \end{cases}$$

for some parameters $c_1 < c_2$

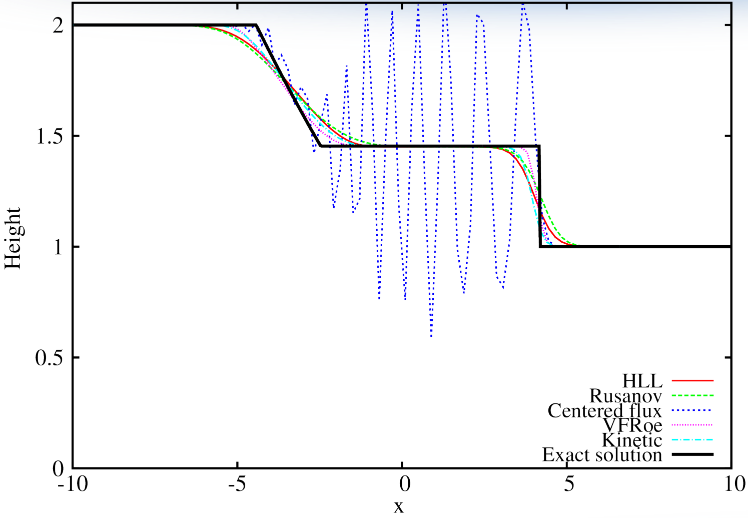

FIGURE: SWE with $u_l = 2$ and $u_r = 1$

1. Introduction
   - Applications
   - Derivation of SWE
   - Properties of SWE and numerical illustration
2. Numerical approximation of SWE
   - Finite volume scheme for Homogenous SWE
   - Finite volume scheme for non Homogenous SWE
   - Kinetic scheme
3. some physical applications derived from SWE
   - A sediment transport model
   - An unsteady mixed flows in closed water pipes model (PFS model)
4. Concluding remarks

- Generality : Cell-centered scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta t_n}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right) + \Delta t_n \int_{t_n}^{t_{n+1}} \int_{m_i} S(t,x)\ dx\ dt$$

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  - Consistency of the flux
  - Stability properties
  - BUT usually$\rightarrow$ bad results close to steady state solution $\rightarrow$ numerical oscillations

$$\frac{\left(F_{i+1/2} - F_{i-1/2}\right)}{\Delta x} + \int_{t_n}^{t_{n+1}} \int_{m_i} S(t,x)\ dx\ dt \neq 0$$

- Generality : Upwinded scheme (USI methods)

$$u_i^{n+1} = u_i^n - \frac{\Delta t_n}{\Delta x}\left(F^-_{i+1/2} - F^+_{i-1/2}\right)$$

with

$$F^-_{i+1/2}(u_i^n, u_{i+1}^n; z_{i+1} - z_i) = F_{i+1/2}(u_i^n, u_{i+1}^n) + S^-_{i+1/2}(u_i^n, u_{i+1}^n, z_{i+1} - z_i)$$

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- Stability : positivity, steady states, discrete entropy inequality
- Generalisation of the Lax-Wendroff theorem
- Noting $D(u) + z = cte$ a steady state, then

Theorem (Katsaounis, Perthame, Simeoni, Applied Mathematics Letters, 04)

A USI scheme is well-balanced iff for all u, v, z_+, z_- such that $D(u) + z_- = D(v) + z_+$, one has

$$F(u,v) - F(u,u) + S^-(u,v,z_+ - z_-) = 0 \text{ and } F(v,v) - F(u,v) + S^+(u,v,z_+ - z_-) = 0$$

It is technically hard to show if a numerical scheme :

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There exist only two scheme able to :

- Hyrostatic reconstruction (Audusse, Bouchut,. . .)
- **Kinetic scheme** (Perthame, Simeoni)

- Generalized Godunov schemes (Leroux, Seguin, . . .)
- VFRoe schemes (Gallouët, Ersoy,. . .)
- Hydrostatic reconstruction (Audusse, Bouchut, . . .)
- Kinetic scheme (Perthame, Simeoni, Ersoy, . . .)
- steady state profil (Greenberg, Leroux, Ersoy)
- Central upwind (Kurganov)
- . . .

Kinetic scheme (c.f. Perthame, Simeoni, Calcolo, 01)

Consider the SWE with a bathymetry term :

$$\left\{\begin{array}{lcl}\partial_t h + \partial_x(hu) & = & 0 \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) & = & -ghZ'(x)\end{array}\right.$$

$$\text{where} \quad \begin{array}{lcl} h(t,x) & : & \text{density} \\ u(t,x) & : & \text{velocity of the water column} \\ Z(x) & : & \text{topography}\end{array}$$

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	$h(t,x)$	:	density
where	$u(t,x)$	:	velocity of the water column
	$Z(x)$	:	topography

Theorem

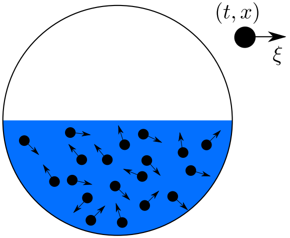
The SW system is strictly hyperbolic for $h > 0$. It admits a mathematical entropy, which is also the physical energy $s(h,u,Z) = \dfrac{hu^2}{2} + \dfrac{gh^2}{2} + ghZ$ which satisfies the entropy inequalities :

$$\partial_t s + \partial_x \left(u \left(s + \frac{gh^2}{2} \right) \right)$$

Moreover, the system admits a family of smooth steady states characterized by

$$\left\{\begin{array}{lcl} hu & = & c_1 \\ \dfrac{u^2}{2} + g(h+Z) & = & c_2 \end{array}\right.$$

As in gas theory ,
Describe the macroscopic behavior from particle motions, here, assumed fictitious by introducing
$$\left\{ \begin{array}{l} \text{a } \chi \text{ density function and} \\ \text{a } \mathcal{M}(t,x,\xi;\chi) \text{ maxwellian function (or a Gibbs equilibrium)} \end{array} \right.$$



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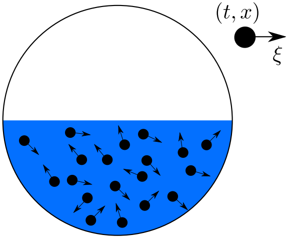
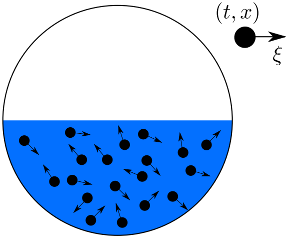


i.e., transform the nonlinear system into a kinetic transport equation on $\mathcal{M}$.

As in gas theory ,
Describe the macroscopic behavior from particle motions, here, assumed fictitious by introducing $\left\{ \begin{array}{l} \text{a } \chi \text{ density function and} \\ \text{a } \mathcal{M}(t,x,\xi;\chi) \text{ maxwellian function (or a Gibbs equilibrium)} \end{array} \right.$



i.e., transform the nonlinear system into a kinetic transport equation on $\mathcal{M}$.
Thus, to be able to define the numerical *macroscopic fluxes* from the microscopic one.

...Faire d'une pierre deux coups...

We introduce

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = \frac{g}{2} \ ,$$

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then we define the Gibbs equilibrium by

$$\mathcal{M}(t,x,\xi) = \sqrt{h(t,x)}\, \chi\left(\frac{\xi - u(t,x)}{\sqrt{h(t,x)}}\right)$$

Principle

Since

$$\chi(\omega) = \chi(-\omega) \geq 0\ ,\ \int_{\mathbb{R}} \chi(\omega) d\omega = 1,\ \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = \frac{g}{2}\ ,$$

and

$$\mathcal{M}(t,x,\xi) = \sqrt{h(t,x)}\, \chi\left(\frac{\xi - u(t,x)}{\sqrt{h(t,x)}}\right)$$

then

Micro-macroscopic relations

$$\begin{aligned} h &= \int_{\mathbb{R}} \mathcal{M}(t,x,\xi)\, d\xi \\ hu &= \int_{\mathbb{R}} \xi \mathcal{M}(t,x,\xi)\, d\xi \\ hu^2 + \frac{gh^2}{2} &= \int_{\mathbb{R}} \xi^2 \mathcal{M}(t,x,\xi)\, d\xi \end{aligned}$$

THE KINETIC FORMULATION

(h, hu) is solution of the SW system if and only if $\mathcal{M}$ satisfy the transport equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\partial_x Z \, \partial_\xi \mathcal{M} = \mathcal{K}(t, x, \xi)$$

where $\mathcal{K}(t, x, \xi)$ is a collision kernel satisfying a.e. (t, x)

$$\int_{\mathbb{R}} \mathcal{K} \, d\xi = 0 \, , \, \int_{\mathbb{R}} \xi \, \mathcal{K} d \, \xi = 0 \, .$$

B. Perthame.
Kinetic formulation of conservation laws.
Oxford University Press.
Oxford Lecture Series in Mathematics and its Applications, Vol 21, 2002.

B. Perthame and C. Simeoni
A kinetic scheme for the Saint-Venant system with a source term.
Calcolo, 38(4) :201–231, 2001.

- Recalling that Z is constant per cell

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Then $\forall (t,x) \in [t_n, t_{n+1}[\times \overset{\circ}{m}_i$

$$Z'(x) = 0$$

- Recalling that Z is constant per cell

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$\Longrightarrow$

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- Recalling that Z is constant per cell

Then $\forall (t,x) \in [t_n, t_{n+1}[\times \overset{\circ}{m}_i$

$$Z'(x) = 0$$

$\Longrightarrow$

$$\left\{ \begin{array}{lcl} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}(t_n, x, \xi) \overset{def}{:=} \sqrt{h(t,x)} \chi \left(\dfrac{\xi - u(t_n, x, \xi)}{\sqrt{h(t,x)}} \right) \end{array} \right.$$

by neglecting the collision kernel.

On $[t_n, t_{n+1}[\times m_i$, we have :

$$\left\{\begin{array}{lcl} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}_i^n(\xi) \end{array}\right.$$

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i.e.

$$f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \xi \frac{\Delta t^n}{\Delta x} \left(\mathcal{M}_{i+\frac{1}{2}}^-(\xi) - \mathcal{M}_{i-\frac{1}{2}}^+(\xi)\right)$$

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where

$$\mathbf{U}_i^{n+1} = \left(\begin{array}{c} A_i^{n+1} \\ Q_i^{n+1} \end{array}\right) \overset{def}{:=} \int_{\mathbb{R}} \left(\begin{array}{c} 1 \\ \xi \end{array}\right) f_i^{n+1}(\xi)\, d\xi$$

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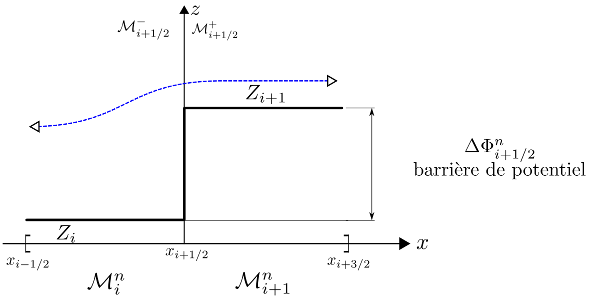
or

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x}\left(F_{i+1/2}^- - F_{i-1/2}^+\right)$$

with

$$F_{i\pm\frac{1}{2}}^{\pm} = \int_{\mathbb{R}} \xi \left(\begin{array}{c} 1 \\ \xi \end{array}\right) \mathcal{M}_{i\pm\frac{1}{2}}^{\pm}(\xi)\, d\xi.$$

$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}_i^n(\xi)}^{\text{positive transmission}} \\ \underbrace{+\,\mathbb{1}_{\{\xi<0,\, \xi^2-2g\Delta\Phi^n_{i+1/2}>0\}}\mathcal{M}^n_{i+1}\left(-\sqrt{\xi^2-2g\Delta\Phi^n_{i+1/2}}\right)}_{\text{negative transmission}}$$



$$\Delta\Phi_{i+1/2} := \Delta Z_{i+1/2} = Z_{i+1} - Z_i$$

The microscopic fluxes

Interpretation : potential bareer

$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}^n_i(\xi)}^{\text{positive transmission}} + \overbrace{\mathbb{1}_{\{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}<0\}}\mathcal{M}^n_i(-\xi)}^{\text{reflection}}$$
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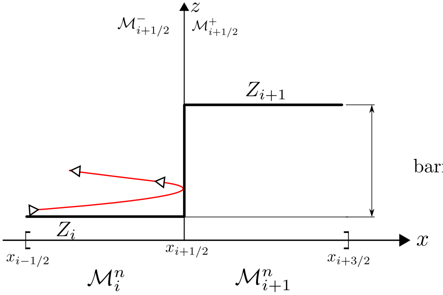

$\Delta\Phi^n_{i+1/2}$
barrière de potentiel

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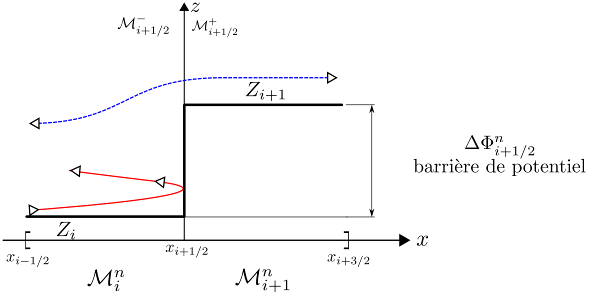

$$\Delta\Phi_{i+1/2} := \Delta Z_{i+1/2} = Z_{i+1} - Z_i$$

Let us recall that we have to define a χ function such that :

$$\chi(\omega) = \chi(-\omega) \geq 0 \; , \; \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = \frac{g}{2} \; ,$$

and $\mathcal{M} = \sqrt{h}\chi\left(\dfrac{\xi - u}{\sqrt{h}}\right)$ satisfies the equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - gZ' \, \partial_\xi \mathcal{M} = 0$$

and

$$\chi \longrightarrow \text{definition of the macroscopic fluxes.}$$

One has

- Conservativity of h holds for every χ.
- Positivity of A holds for every χ but for numerical purpose only if $\mathrm{supp}\chi$ is compact to get a CFL condition.

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strongly depend on the choice of the χ function.

In the following, we only focus on discrete equilibrium.

Strategy

Set χ such that $\mathcal{M}(t,x,\xi;\chi)$ is the still water steady state solution of :

$$\xi \cdot \partial_x \mathcal{M} - gZ' \, \partial_\xi \mathcal{M} = 0.$$

Then

$$w\chi(w) + \left\{2g - w^2\right\} \chi'(w) = 0$$

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As a consequence, this function is the only possible choice such that $M(t,x,\xi) = \sqrt{h}\chi\left(\dfrac{\xi - u}{\sqrt{h}}\right)$ satisfies the equation

$$\xi \cdot \partial_x \mathcal{M} - gZ' \, \partial_\xi \mathcal{M} = 0.$$

on any still water steady states,

$$u(t,x) = 0, \quad h(t,x) + Z(x) = H, \ \forall t \geqslant 0 \ .$$

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on any still water steady states,

$$u(t, x) = 0, \quad h(t, x) + Z(x) = H, \ \forall t \geqslant 0 \ .$$

Moreover, one can interpret, such a solution, as the minimum of the kinetic energy

$$E(f) = \int_{\mathbb{R}} \left(\frac{\xi^2}{2} f(\xi) + \frac{\pi^2 g^2}{6} f^3(\xi) + gZf(\xi) d\xi \right)$$

With this χ function, one has

THEOREM

Under the CFL condition

$$\Delta t_n \max\left(|u_i^n| + \sqrt{2gh_i^n}\right) \leqslant \Delta x \ ,$$

the following assertions hold :

1. *Positivity of h,*
2. *Conservativity of h,*
3. *Discrete still water steady states are preserved,*
4. *Discrete in-entropy inequalities are satisfied,*
5. *Drying and flooding phenomenon are naturally obtained*

As a conclusion, EVEN IF THE INTEGRAL COMPUTATION ARE NOT EXPLICIT, this kinetic scheme have all the necessary properties to get consistency, stability, convergence.

With this χ function, one has

Theorem

Under the CFL condition

$$\Delta t_n \max \left(|u_i^n| + \sqrt{2gh_i^n} \right) \leqslant \Delta x \ ,$$

the following assertions hold :

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5. *Drying and flooding phenomenon are naturally obtained*

In practice, I prefer to use the following simplest χ-function

$$\chi(\omega) = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(\omega)$$

with

- integral computation are explicit and easy
- loosing the property 3 and a priori 4 (even if the numerical results provides good agreements with test cases)

1. Introduction
   - Applications
   - Derivation of SWE
   - Properties of SWE and numerical illustration
2. Numerical approximation of SWE
   - Finite volume scheme for Homogenous SWE
   - Finite volume scheme for non Homogenous SWE
   - Kinetic scheme
3. some physical applications derived from SWE
   - A sediment transport model
   - An unsteady mixed flows in closed water pipes model (PFS model)
4. Concluding remarks

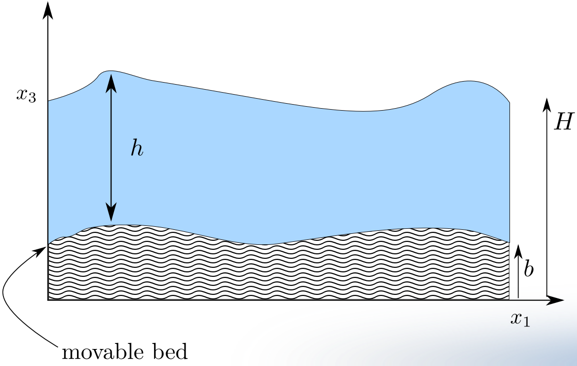
Saint-Venant equations for the hydrodynamic part :

$$\begin{cases} \partial_t h + \mathrm{div}(q) = 0, \\ \partial_t q + \mathrm{div}\left(\dfrac{q \otimes q}{h}\right) + \nabla\left(g\dfrac{h^2}{2}\right) = -gh\nabla b \end{cases} \tag{1}$$

$+$

a bedload transport equation for the morphodynamic part :

$$\partial_t b + \xi \mathrm{div}(q_b(h,q)) = 0 \tag{2}$$



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with

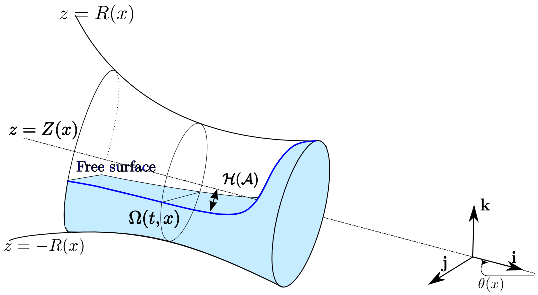
- h : water height,
- $q = hu$: water discharge,
- q_b : sediment discharge (empirical law : [MPM48], [G81]),
- $\xi = 1/(1-\psi)$: porosity coefficient.

M. Ersoy,
Modélisation, analyse mathématique et numérique de divers écoulements compressibles ou incompressibles en couche mince,
Ph.D University of Savoie (France), 2010.

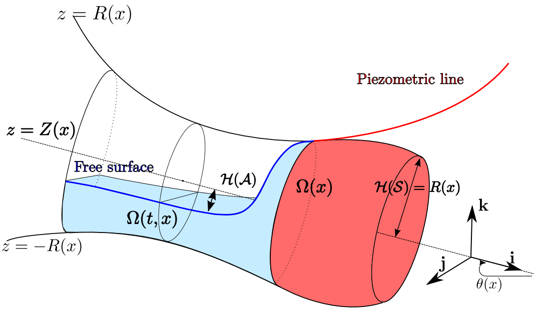
E. Meyer-Peter and R. Müller,
Formula for bed-load transport,
Rep. 2nd Meet. Int. Assoc. Hydraul. Struct. Res., 39–64, 1948.

A.J. Grass,
Sediment transport by waves and currents,
SERC London Cent. Mar. Technol. Report No. FL29 , 1981.

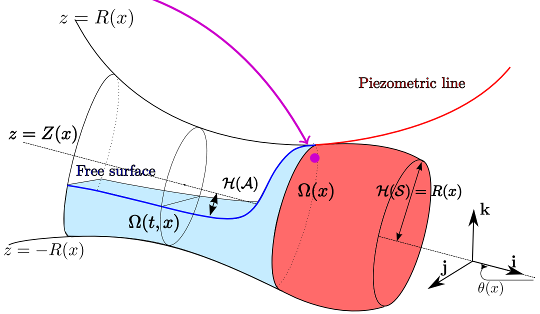
- Free surface area (SL)
  sections are not completely filled and the flow is incompressible. . .

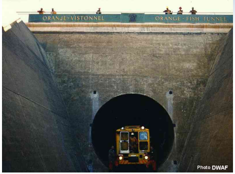


- Free surface area (SL)
  sections are not completely filled and the flow is incompressible...
- Pressurized area (CH)
  sections are non completely filled and the flow is compressible...



- Free surface area (SL)
  sections are not completely filled and the flow is incompressible. . .
- Pressurized area (CH)
  sections are non completely filled and the flow is compressible. . .
- Transition point





Orange-Fish tunnel

Sewers . . . in Paris

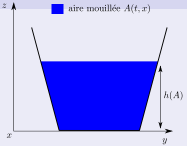
Forced pipe

problems . . . at Minnesota
http://www.sewerhistory.org/grfx/misc/disaster.htm

Generally

Saint-Venant equations :

$$\begin{cases} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(A) \right) = 0 \end{cases}$$



with

$A(t,x)$	:	wet area
$Q(t,x)$	:	discharge
$I_1(A)$	:	hydrostatic pressure
g	:	gravity

Advantage

- Conservative formulation $\longrightarrow$ Easy numerical implementation

Hamam and McCorquodale (82), Trieu Dong (91), Musandji Fuamba (02), Vasconcelos *et al* (06)

GENERALLY

Allievi equations :

$$\begin{cases} \partial_t p + \dfrac{c^2}{gS}\partial_x Q = 0, \\ \partial_t Q + gS\partial_x p = 0 \end{cases}$$

with

$p(t,x)$	:	pressure
$Q(t,x)$	:	discharge
$c(t,x)$	:	sound speed
$S(x)$	:	section

Advantage

- Compressibility of water is taking into account $\Longrightarrow$ Sub-atmospheric flows and over-pressurized flows are well computed

Drawback

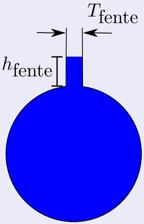
- Non conservative formulation $\Longrightarrow$ Cannot be, at least easily, coupled to Saint-Venant equations

Winckler (93), Blommaert (00)

Generally

Saint-Venant with Preissmann slot artifact :

$$\begin{cases} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(A) \right) = 0 \end{cases}$$



Advantage

- Only one model for two types of flows.

Drawbacks

- Incompressible Fluid $\Longrightarrow$ Water hammer not well computed
- Pressurized sound speed $\simeq \sqrt{S/T_{\text{fente}}}$ $\Longrightarrow$ adjustment of T_{fente}
- Depression $\Longrightarrow$ seen as a free surface state

Preissmann (61), Cunge *et al.* (65), Baines *et al.* (92), Garcia-Navarro *et al.* (94), Capart *et al.* (97), Tseng (99)

Our goal :

- Use Saint-Venant equations for free surface flows

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- Write a pressurized model
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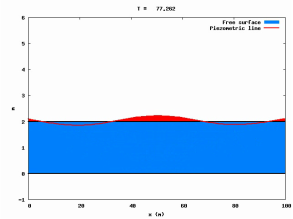
- Use Saint-Venant equations for free surface flows
- Write a pressurized model
  - which takes into account the compressibility of water
  - which takes into account the depression
  - similar to Saint-Venant equations
- Get one model for mixed flows

To be able to simulate, for instance :



Ersoy et al. *A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.*, Int. J. On Finite Volumes, 6(2), pp 1-47, 2009.

$$
\begin{array}{rcl}
\rho_0 \mathrm{div}(\mathbf{U}) & = & 0 \\
\rho_0(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}) + \nabla p & = & \rho_0 F
\end{array}
$$

Method :

1. Write Euler equations in curvilinear coordinates.
2. Write equations in non-dimensional form using the small parameter $\epsilon = H/L$ and takes $\epsilon = 0$.
3. Section averaging $\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. Introduce $A_{sl}(t,x)$: wet area, $Q_{sl}(t,x)$ discharge given by :

$$
A_{sl}(t,x) = \int_{\Omega(t,x)} dydz, \quad Q_{sl}(t,x) = A_{sl}(t,x)u(t,x)
$$

$$
u(t,x) = \frac{1}{A_{sl}(t,x)} \int_{\Omega(t,x)} U(t,x)\; dydz
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J.-F. Gerbeau, B. Perthame
Derivation of viscous Saint-Venant System for Laminar Shallow Water ; Numerical Validation.
Discrete and Continuous Dynamical Systems, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

F. Marche
Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects.
European Journal of Mechanic B/Fluid, 26 (2007), 49–63.

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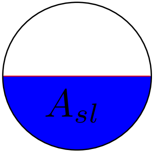

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p_{sl} & = & g I_1(x, A_{sl}) \cos\theta & : \text{hydrostatic pressure law} \\
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with

$$p_{sl} = gI_1(x, A_{sl})\cos\theta \quad \text{: hydrostatic pressure law}$$

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$$K = \frac{1}{K_s^2 R_h(A_{sl})^{4/3}} \quad \text{: Manning-Strickler law}$$

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{U}) = 0$$
$$\partial_t(\rho \mathbf{U}) + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{F}$$

with

$$p = p_a + \frac{\rho - \rho_0}{c^2} \text{ with } c \text{ sound speed}$$

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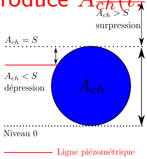

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To be coupled

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$$E = \begin{cases} 1 & \text{if the flow is pressurized (CH),} \\ 0 & \text{if the flow is free surface (SL)} \end{cases}$$

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$$\begin{array}{rcll} A & = & \dfrac{\bar{\rho}}{\rho_0}\mathbf{S} = \begin{cases} \mathbf{S}(A_{sl}, 0) = A_{sl} & \text{if} \quad \text{SL} \\ \dfrac{\bar{\rho}}{\rho_0}\mathbf{S}(A_{sl}, 1) = A_{ch} & \text{if} \quad \text{CH} \end{cases} & : \quad \text{the « mixed » variable} \\ Q & = & Au & : \quad \text{the discharge} \end{array}$$

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Continuity of **S** at transition point

- Continuity of **S** $\Longrightarrow$ continuity of p at transition point

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- Similar construction for the pressure source term :
$$Pr(x,A,E) = c^2\left(\frac{A}{\mathbf{S}} - 1\right)\frac{d\,S}{dx} + gI_2(x,\mathbf{S})\cos\theta$$

$$
\left\{
\begin{array}{ll}
\partial_t(A) + \partial_x(Q) & = 0 \\[2ex]
\partial_t(Q) + \partial_x \left(\dfrac{Q^2}{A} + p(x,A,E) \right) & = -g\, A\, \dfrac{d}{dx} Z(x) \\
 & +Pr(x,A,E) \\[2ex]
 & -G(x,A,E) \\[2ex]
 & -g\, K(x, \mathbf{S})\, \dfrac{Q|Q|}{A}
\end{array}
\right.
$$

Ersoy et al., *A model for unsteady mixed flows in non uniform closed water pipes.*, SCIENCE CHINA Mathematics, 55(1) :1–26, 2012.

MATHEMATICAL PROPERTIES

- The **PFS** system is strictly hyperbolic for $A(t,x) > 0$.
- For regular solutions, the mean speed $u = Q/A$ verifies

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + c^2 \ln(A/S) + g\,\mathcal{H}(S)\cos\theta + g\,Z \right) = -g\,K(x, \mathbf{S})\,u\,|u|$$

and for $u = 0$, we have :

$$c^2 \ln(A/\mathbf{S}) + g\,\mathcal{H}(\mathbf{S})\cos\theta + g\,Z = cte$$

where $\mathcal{H}(\mathbf{S})$ is the physical water height.

- There exists a mathematical entropy

$$E(A,Q,S) = \frac{Q^2}{2A} + c^2 A \ln(A/\mathbf{S}) + c^2 S + g\overline{z}(x,\mathbf{S})\cos\theta + gAZ$$

which satisfies

$$\partial_t E + \partial_x \left(E\,u + p(x,A,E)\,u \right) = -g\,A\,K(x,\mathbf{S})\,u^2\,|u| \leqslant 0$$

The PFS model have complexe source terms which makes its analysis hard !
We have developped two numerical scheme :

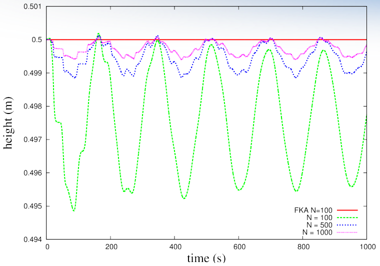
- VFRoe (**Ersoy** et al., IJFV, 09)
- Kinetic (**Ersoy** et al., JSC, 11)

Nevertheless (for both), one can analytically check

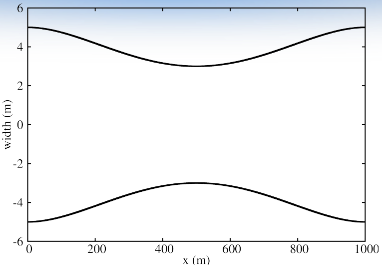
1. Positivity of A,
2. Conservativity of A,
3. Drying and flooding phenomenon are naturally obtained
4. Discrete still water steady states are preserved exactly (only for VFRoe scheme)

and numerically check that

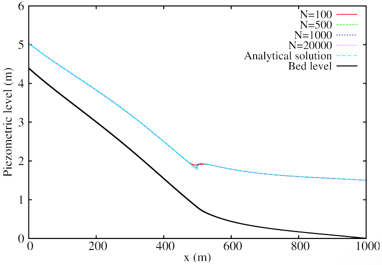
1. Discrete still water steady states are $\approx$ preserved,
2. Discrete in-entropy inequalities are *a priori* satisfied.



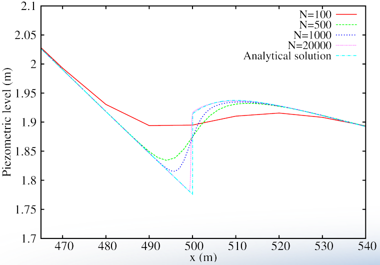
VFRoe : a still water steady states



Kinetic : variation of section



Kinetic : a transcritical steady solution with shock



Kinetic : a transcritical steady solution with shock

THE KINETIC FORMULATION

(A, Q) is solution of the PFS system if and only if $\mathcal{M}$ satisfy the transport equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\Phi\, \partial_\xi \mathcal{M} = \mathcal{K}(t, x, \xi)$$

where $\mathcal{K}(t, x, \xi)$ is a collision kernel satisfying a.e. (t, x)

$$\int_{\mathbb{R}} \mathcal{K}\, d\xi = 0\ ,\ \int_{\mathbb{R}} \xi\, \mathcal{K} d\,\xi = 0$$

and Φ are the source terms.

B. Perthame.
Kinetic formulation of conservation laws.
Oxford University Press.
Oxford Lecture Series in Mathematics and its Applications, Vol 21, 2002.

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and Φ are the source terms.

General form of the source terms :

$$\Phi = \overbrace{\frac{d}{dx} Z}^{\text{conservative}} + \overbrace{\mathbf{B} \cdot \frac{d}{dx} \mathbf{W}}^{\text{non conservative}} + \overbrace{K \frac{Q|Q|}{A^2}}^{\text{friction}}$$

with $\mathbf{W} = (Z, S, \cos\theta)$

- conservative term : classical upwind
- non conservative term : mid point rule (DLM, 95)
- friction : dynamic topography (Ersoy, Ph.D.)

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- nevertheless, there are still several problems (even in the 1D case)
  - for conservative source terms (typically Z') : scheme preserving all steady states ?
  - for non conservative (as in the PFS model) : scheme preserving still water steady states exactly as well as an entropic scheme ?

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Thank you

for your

attention