Existence and stability results for some compressible primitive equations

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Outline of the talk

1. Introduction
2. Main results
   - An existence result for the 2D-CPEs
   - A stability result for the 3D-CPEs
3. Perspectives

Outline

Context

Navier-Stokes equations (NSEs) or Euler equations (EEs) on
$\Omega = \{(x,y) \in \mathbb{R}^3; H \ll L\}$ "thin layer domain"

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Hydrostatic approximation (asymptotic analysis with $\varepsilon = H/L = W/V \ll 1$ and rescaling $\tilde{x} = x/L$, $\tilde{y} = y/H$, $\tilde{u} = u/U$ $\tilde{w} = w/W$)$\longrightarrow$ Primitive equations (PEs)

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Averaged PEs with respect to depth or altitude $y \longrightarrow$ Saint-Venant Equations (SVEs)

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Geophysical Fluid Dynamics.
2nd Edition, Springer-Verlag, New-York, 1987.

J.-F Gerbeau and B. Perthame
Derivation of viscous Saint-Venant system for laminar shallow water ; numerical validation.
Discrete Contin. Dyn. Syst. Ser. B, 1(1), 2001.

Atmosphere dynamic

- Dynamic :
  - Compressible fluid
  - Small vertical extension with respect to horizontal
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  - Density stratified

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- Modeling (neglecting phenomena such as the evaporation and solar heating) :
  Compressible Navier-Stokes equations

  Hydrostatic approximation $\longrightarrow$ compressible primitive equations (CPEs)

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\rho + \rho \mathrm{div}\mathbf{U} & = & 0 \\ \rho\dfrac{d}{dt}\mathbf{u} + \nabla_x p & = & \mathrm{div}_x(\sigma_x) + f \\ \partial_t(\rho v) + \mathrm{div}(\rho \mathbf{U} v) + \partial_y p(\rho) & = & -\rho g + \mathrm{div}_y(\sigma_y) \\ p(\rho) & = & c^2\,\rho \end{array}\right.$$

with $\dfrac{d}{dt} := \partial_t + \mathbf{u}\cdot\nabla_x + v\partial_y$

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M. Ersoy and T. Ngom
Existence of a global weak solution to one model of Compressible Primitive Equations.
Submitted, 2010.

M. Ersoy, T. Ngom and M. Sy
Compressible primitive equations : formal derivation and stability of weak solutions.
Nonlinearity, 24(1), pp 79-96, 2011.

Framework

Main difference with respect to the classical viscous term found in the literature (see, for instance, Temam and Ziane [TZ04]) : here

viscosities depend on the density and are anisotropic.

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Find a change of variables (in the same spirit of Lions *et al* [LTW92]) to get a similar model as in [GK05], that is to say, change the hydrostatic equation

$$c^2\partial_y\rho = -g\,\rho \text{ into } \partial_z\xi = 0.$$

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A useful change of variables [EN10]

Let us consider the following two dimensional problem :

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\rho + \rho\,\mathrm{div}\mathbf{U} & = & 0 \\ \rho\dfrac{d}{dt}\mathbf{u} + c^2\partial_x\rho & = & \partial_x(\nu_1(t,x,y)\partial_x u) + \partial_y(\nu_2(t,x,y)\partial_y u) \\ c^2\partial_y\rho = -g\rho & & \end{array}\right.$$

with $\mathbf{U} = (\mathbf{u}, v) \in \mathbb{R}^2$
or equivalently, in conservative form :

$$\left\{\begin{array}{rcl} \partial_t\rho + \partial_x(\rho\mathbf{u}) + \partial_y(\rho v) & = & 0 \\ \partial_t(\rho\mathbf{u}) + \partial_x(\rho\mathbf{u}^2) + \partial_y(\rho\mathbf{u}v) + c^2\partial_x\rho & = & \partial_x(\nu_1(t,x,y)\partial_x\mathbf{u}) \\ & & +\partial_y(\nu_2(t,x,y)\partial_y\mathbf{u}) \\ c^2\partial_y\rho & = & -g\rho \end{array}\right.$$

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Then,

- Set $\rho = \xi(t,x)e^{-\frac{g}{c^2}y}$, $\nu_1(t,x,y) = \bar{\nu_1}e^{-\frac{g}{c^2}y}$, $\nu_2(t,x,y) = \bar{\nu_2}e^{\frac{g}{c^2}y}$, $(\bar{\nu_1},\bar{\nu_2}) \in \mathbb{R}^2$ and multiply by $e^{\frac{g}{c^2}y}$

Let us consider the following two dimensional problem :

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- Set $\partial_z\cdot = e^{\frac{g}{c^2}y}\partial_y\cdot$ and $w = e^{-\frac{g}{c^2}y}v$

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Finally, we get :

$$\left\{\begin{array}{rcl} \partial_t\xi + \partial_x(\xi\mathbf{u}) + \partial_z(\xi\mathbf{u}) & = & 0 \\ \partial_t(\xi\mathbf{u}) + \partial_x(\xi\mathbf{u}^2) + \partial_z(\xi\mathbf{u}w) + c^2\partial_x\xi & = & \overline{\nu_1}\partial_{xx}\mathbf{u} \\ & & +\overline{\nu_2}\partial_{zz}\mathbf{u} \\ \partial_z\xi = 0 & & \end{array}\right.$$

or equivalently, in non-conservative form :

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\xi + \xi\mathrm{div}\mathbf{U} & = & 0 \\ \xi\dfrac{d}{dt}\mathbf{u} + c^2\partial_x\xi & = & \overline{\nu_1}\partial_{xx}\mathbf{u} + \overline{\nu_2}\partial_{zz}\mathbf{u}\partial_z\xi = 0 \end{array}\right.$$

with

- $\mathbf{U} := (\mathbf{u}, w)$,
- $\dfrac{D}{Dt} := \partial_t + \mathbf{U}\cdot\nabla$,
- $\nabla := (\partial_x, \partial_z)^t$,
- $\mathrm{div} := \partial_x + \partial_z$.

and corresponds exactly to the model studied by [GK05] : existence of weak solutions global in time for the model with $(\rho, \mathbf{u})$ is then a straightforward consequence.

OUTLINE

THE 3D-CPEs

Let us consider the following model posed on $\Omega = \{(x,y);\, x \in \mathcal{T}^2,\, 0 < y < 1\}$:

$$\begin{cases} \dfrac{d}{dt}\rho + \rho \mathrm{div}\mathbf{U} = 0, \\ \rho\dfrac{d}{dt}\mathbf{u} + \nabla_x p = 2\mathrm{div}_x\left(\nu_1(t,x,y)D_x(\mathbf{u})\right) + \partial_y\left(\nu_2(t,x,y)\partial_y \mathbf{u}\right), \\ \partial_y p = -g\rho, \\ p(\rho) = c^2\,\rho \end{cases}$$

with

$$\begin{aligned} &\text{periodic conditions on } \partial\Omega_x, \\ &v_{|y=0} = v_{|y=H} = 0, \\ &\partial_y \mathbf{u}_{|y=0} = \partial_y \mathbf{u}_{|y=H} = 0. \end{aligned}$$

and

$$\begin{aligned} \mathbf{u}(0,x,y) &= \mathbf{u}_0(x,y), \\ \rho(0,x,y) &= \xi_0(x)e^{-g/c^2 y} \end{aligned}$$

where

$$0 \leqslant \xi_0(x) \leqslant M < +\infty.$$

Energy estimates ? ? ?

Let us multiply the previous system by **U**, we get :

$$\frac{d}{dt}\int_\Omega (\rho|\mathbf{u}|^2 + \rho\ln\rho - \rho + 1)\,dxdy + \int_\Omega 2\nu_1|D_x(\mathbf{u})|^2 + \nu_2|\partial_y^2\mathbf{u}|\,dxdy + \int_\Omega \rho g v\,dxdy$$

where $\int_\Omega \rho g v\,dxdy >? < 0$? ? ?.

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Could we simply multiply by **u** instead of **U** ?

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where $\int_\Omega \rho g v\,dxdy >? < 0$? ? ?.

No, it is not enough to get useful estimates for stability of solutions.

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where $\int_\Omega \rho g v\,dxdy$ $>?<0$? ? ?.

However, if the rhs of the last is zero : from the mass equation, we have

$$\partial_{zz}w=\frac{1}{\xi}\mathrm{div}_x(\xi\partial_z\mathbf{u})$$

a crucial information to get additional estimates.
Consequently, we systematically perform the previous change of variables, i.e. changes $(\rho,\mathbf{u},v)$ in $(\xi,\mathbf{u},w)$.

Viscosities ? ? ?

If we choose the previous viscosities, we get :

$$\left\{\begin{array}{l}\dfrac{d}{dt}\xi + \xi \mathsf{div}\mathbf{U} = 0,\\ \xi\dfrac{d}{dt}\mathbf{u} + \nabla_x p = \overline{\nu_1}\Delta_x \mathbf{u} + \overline{\nu_2}\partial_{yy}\mathbf{u},\\ \partial_z \xi = 0\end{array}\right.$$

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- energy estimates OK !

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- energy estimates OK !
- No way to establish results [4] : Lagrangian coordinates approach as in [GK05] fails.

4. up to our knowledge

VISCOSITIES ? ? ?

Choose $\nu_1(t,x,y) = \bar{\nu}_1\rho(t,x,y)$ and $\nu_2(t,x,y) = \bar{\nu}_2\rho(t,x,y)e^{2y}$ with $\bar{\nu}_i > 0$,, we get :

$$\begin{cases} \dfrac{d}{dt}\xi + \xi(\mathrm{div}_x\mathbf{u} + \partial_z w) = 0, \\ \xi\dfrac{d}{dt}\mathbf{u} + c^2\nabla_x\xi = 2\bar{\nu}_1\mathrm{div}_x\left(\xi D_x(\mathbf{u})\right) + \bar{\nu}_2\partial_z\left(\xi\nu_2(t,x,z)\partial_z\mathbf{u}\right), \\ \partial_z\xi = 0, \\ p(\xi) = c^2\,\xi \end{cases} \tag{1}$$

Then,

- Existence ? ? ?
- Stability of weak solutions : Yes ! ! ! by adding a regularizing term to equations in order to pass to the limit in the non-linear term $\xi\mathbf{u}^2$ (BD-entropy).

With these settings

Multiply by **U**, the energy reads :

$$\frac{d}{dt}\int_{\Omega'}\left(\xi\frac{\mathbf{u}^2}{2}+(\xi\ln\xi-\xi+1)\right)dxdz+\int_{\Omega'}\xi(2\bar{\nu}_1|D_x(\mathbf{u})|^2+\bar{\nu}_2|\partial_z\mathbf{u}|^2)\,dxdz$$
$$+r\int_{\Omega'}\xi|\mathbf{u}|^3\,dxdz\leqslant 0 \qquad (2)$$

which provides the uniform estimates :

$$\sqrt{\xi}\mathbf{u} \text{ is bounded in } L^\infty(0,T;(L^2(\Omega^{'}))^2),$$
$$\xi^{\frac{1}{3}}\mathbf{u} \text{ is bounded in } L^3(0,T;(L^3(\Omega^{'}))^2),$$
$$\sqrt{\xi}\partial_z\mathbf{u} \text{ is bounded in } L^2(0,T;(L^2(\Omega^{'}))^2),$$
$$\sqrt{\xi}D_x(\mathbf{u}) \text{ is bounded in } L^2(0,T;(L^2(\Omega^{'}))^{2\times 2}),$$
$$\xi\ln\xi-\xi+1 \text{ is bounded in } L^\infty(0,T;L^1(\Omega^{'})).$$

With these settings

Following BD the strong convergence of $\sqrt{\xi}\mathbf{u}$ required to pass to the limit in the non linear term $\xi\mathbf{u}\otimes\mathbf{u}$ is obtained by the BD entropy :
Take the gradient of the mass equation, multiply by $2\bar{\nu}_1$, write the term $\nabla_x\xi$ as $\xi\nabla_x\ln\xi$, combine with the momentum equations, to get the entropy inequality :

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega'}\left(\xi|\mathbf{u}+2\bar{\nu}_1\nabla_x\ln\xi|^2+2(\xi\log\xi-\xi+1)\right)\,dxdz$$
$$+\int_{\Omega'}2\bar{\nu}_1\xi|\partial_z w|^2+2\bar{\nu}_1\xi|A_x(u)|^2+\bar{\nu}_2\xi|\partial_z\mathbf{u}|^2\,dxdz$$
$$+\int_{\Omega'}r\xi|\mathbf{u}|^3+2\bar{\nu}_1 r|\mathbf{u}|\mathbf{u}\nabla_x\xi+8\bar{\nu}_1|\nabla_x\sqrt{\xi}|^2\,dxdz=0. \quad (3)$$

which gives the following estimates :

$$\nabla\sqrt{\xi} \text{ is bounded in } L^\infty(0,T;(L^2(\Omega^{'}))^3),$$
$$\sqrt{\xi}\partial_z w \text{ is bounded in } L^2(0,T;L^2(\Omega^{'})),$$
$$\sqrt{\xi}A_x(\mathbf{u}) \text{ is bounded in } L^2(0,T;(L^2(\Omega^{'}))^{2\times 2})\,.$$

With these settings

Define the set of function $\rho \in \mathcal{PE}(\mathbf{u}, v; y, \rho_0)$ which satisfy

$$\begin{array}{ll} \rho \in L^\infty(0,T;L^3(\Omega)), & \sqrt{\rho} \in L^\infty(0,T;H^1(\Omega)), \\ \sqrt{\rho}\mathbf{u} \in L^2(0,T;(L^2(\Omega))^2), & \sqrt{\rho}v \in L^\infty(0,T;L^2(\Omega)), \\ \sqrt{\rho}D_x(\mathbf{u}) \in L^2(0,T;(L^2(\Omega))^{2\times 2}), & \sqrt{\rho}\partial_y v \in L^2(0,T;L^2(\Omega)), \\ \nabla\sqrt{\rho} \in L^2(0,T;(L^2(\Omega))^3) & \end{array}$$

with $\rho \geqslant 0$ and where $(\rho, \sqrt{\rho}\mathbf{u}, \sqrt{\rho}v)$ satisfies :

$$\left\{ \begin{array}{l} \partial_t \rho + \mathrm{div}_x(\sqrt{\rho}\sqrt{\rho}\mathbf{u}) + \partial_y(\sqrt{\rho}\sqrt{\rho}v) = 0, \\ \rho_{t=0} = \rho_0. \end{array} \right.$$

With these settings

Define the integral operators, for any smooth test function φ with compact support such as $\varphi(T,x,y)=0$ and $\varphi_0=\varphi_{t=0}$:

$$\begin{aligned}\mathcal{A}(\rho,\mathbf{u},v;\varphi,dy) \quad = \quad & -\int_0^T\int_\Omega \rho\mathbf{u}\partial_t\varphi\,dxdydt \\ & +\int_0^T\int_\Omega (2\nu_1(t,x,y)\rho D_x(\mathbf{u})-\rho\mathbf{u}\otimes\mathbf{u}):\nabla_x\varphi\,dxdydt \\ & +\int_0^T\int_\Omega r\rho|\mathbf{u}|\mathbf{u}\varphi\,dxdydt-\int_0^T\int_\Omega \rho\mathrm{div}(\varphi)\,dxdydt \\ & -\int_0^T\int_\Omega \mathbf{u}\partial_y(\nu_2(t,x,y)\partial_y\varphi)\,dxdydt \\ & -\int_0^T\int_\Omega \rho v\mathbf{u}\partial_y\varphi\,dxdydt\end{aligned}$$

$$\mathcal{B}(\rho,\mathbf{u},v;\varphi,dy)=\int_0^T\int_\Omega \rho v\varphi\,dxdydt$$

and

$$\mathcal{C}(\rho,\mathbf{u};\varphi,dy)=\int_\Omega \rho_{|t=0}\mathbf{u}_{|t=0}\varphi_0\,dxdy$$

Definition

A weak solution of 3D-CPEs on $[0,T]\times\Omega$, with boundary conditions and initial conditions, is a collection of functions $(\rho,\mathbf{u},v)$ such as $\rho\in\mathcal{PE}(\mathbf{u},v;y,\rho_0)$ and the following equality holds for all smooth test function φ with compact support such as $\varphi(T,x,y)=0$ and $\varphi_0=\varphi_{t=0}$:

$$\mathcal{A}(\rho,\mathbf{u},v;\varphi,dy)+\mathcal{B}(\rho,\mathbf{u},v;\varphi,dy)=\mathcal{C}(\rho,\mathbf{u};\varphi,dy)\ .$$

Theorem

Let $(\rho_n, \mathbf{u}_n, v_n)$ *be a sequence of weak solutions of 3D-CPEs, with boundary conditions and initial conditions, satisfying entropy inequalities* (2) *and* (3) *such as*

$$\rho_n \geqslant 0, \quad \rho_0^n \to \rho_0 \;\; in \;\; L^1(\Omega), \quad \rho_0^n \mathbf{u}_0^n \to \rho_0 \mathbf{u}_0 \;\; in \;\; L^1(\Omega).$$

Then, up to a subsequence,

- ρ_n *converges strongly in* $\mathcal{C}^0(0, T; L^{3/2}(\Omega))$,
- $\sqrt{\rho_n}\mathbf{u}_n$ *converges strongly in* $L^2(0, T; (L^{3/2}(\Omega))^2)$,
- $\rho_n u_n$ *converges strongly in* $L^1(0, T; (L^1(\Omega))^2)$ *for all* $T > 0$,
- $(\rho_n, \sqrt{\rho_n}\mathbf{u}_n, \sqrt{\rho_n}v_n)$ *converges to a weak solution of 3D-CPEs,*
- $(\rho_n, \mathbf{u}_n, v_n)$ *satisfies the energy inequality* (2)*, the entropy inequality* (3) *and converges to a weak solution of 3D-CPEs.*

To show the compactness of sequences $(\xi_n, \mathbf{u}_n, w_n)$ in appropriate space function we follow the work of Mellet *et al.* [MV07] :

1. show the convergence of the sequence $\sqrt{\xi_n}$,
2. we seek bounds of $\sqrt{\xi_n}\mathbf{u}_n$ and $\sqrt{\xi_n} w_n$,
3. prove the convergence of $\xi_n \mathbf{u}_n$,
4. prove the convergence of $\sqrt{\xi_n}\mathbf{u}_n$.

which ends the proof.

A. Mellet and A. Vasseur

On the barotropic compressible Navier-Stokes equations.

Comm. Partial Differential Equations, 32(1-3), pp 431–452, 2007.

Outline

- Show the existence or stability of weak solutions for the 3D-CPEs with $\nu_1 = \overline{\nu_1} e^{-\frac{g}{c^2} y}$ and $\nu_2 = \overline{\nu_2} e^{\frac{g}{c^2} y}$,
- Show the existence of weak solutions for the presented 3D-CPEs.

Thank you

for your

attention

One more thing

Main steps [LTW92, TZ04] :

Equations are

$$\begin{cases} \rho \dfrac{d}{dt}\mathbf{u} + \nabla_x p = \mu \Delta_x \mathbf{u} + \nu \partial_y^2 \mathbf{u}, \\ \partial_y p = -g\rho, \quad p = c^2\,\rho \\ \dfrac{d}{dt}\rho + \rho \mathrm{div}\mathbf{U} = 0, \\ c_p \dfrac{D}{Dt}\mathcal{T} - \dfrac{1}{\rho}\dfrac{D}{Dt}p = Q_{\mathcal{T}}, \\ \dfrac{D}{Dt}q = Q_q \end{cases}$$

J.L. Lions and R. Temam and S. Wang
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R. Temam and M. Ziane
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Ideas :

- Use the pressure as a vertical coordinate.

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- Mass equation is changed into incompressible one : Leray's results are available.

return

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Main steps [GK05] : 2D-CPEs

Equations are

$$\begin{cases} \dfrac{d}{dt}\xi + \xi(\partial_x u + \partial_z w) = 0, \\ \rho\dfrac{d}{dt}\mathbf{u} + \partial_x \xi = \Delta \mathbf{u}, \\ \partial_z \xi = 0. \end{cases}$$

with $\dfrac{d}{dt} := \partial_t + \mathbf{u} \cdot \nabla_x + v\partial_z$

B. V. Gatapov and A. V. Kazhikhov

Existence of a global solution to one model problem of atmosphere dynamics

Siberian Mathematical Journal, 46(5), pp 805-812, 2005.

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- Write mean-oscillation equations and apply a Schauder fixed point theorem

return

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Notations

- $x = (x_1, x_2)$ horizontal and y vertical coordinate,
- $\mathbf{U} = (\mathbf{u} = (u_1, u_2), v)$ velocity vector (horizontal and vertical component),
- ρ density,
- p barotropic pressure,
- g gravity constant,
- c^2 usually set to $\mathcal{RT}$ where $\mathcal{R}$ is the specific gas constant for the air and $\mathcal{T}$ the temperature,
- $\mathrm{div}_x := \partial_{x_1} + \partial_{x_2}$, $D_x = (\nabla_x + \nabla_x^t)/2$,
- $\nu_1(t, x, y) \neq \nu_2(t, x, y)$ represent the anisotropic pair of viscosity depending on the density ρ,
- $\dfrac{D}{Dt} := \partial_t + \mathbf{U} \cdot \nabla$,
- $\dfrac{d}{dt} := \partial_t + \mathbf{u} \cdot \nabla_x + v\partial_y$,
- $2D_x(\mathbf{u}) = \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} = \left(\partial_{x_i} \mathbf{u}_j + \partial_{x_j} \mathbf{u}_i\right)_{1 \leqslant i,j \leqslant 2}$.