# A kinetic scheme for air entrainment in transient flows: a two-layer approach.

**Mehmet Ersoy** [1]
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Dynamics and modeling of complex networks
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Outline of the talk

Outline

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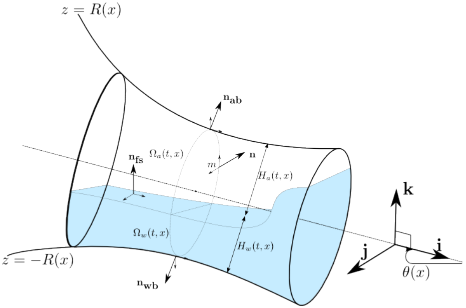
1. Physical and mathematical motivations
   - Air entrainement
   - Previous works
2. The two layers or two-fluids model
   - Fluid Layer : incompressible Euler's Equations
   - Air Layer : compressible Euler's Equations
   - The two-layer model
   - Properties
3. Numerical approximation
   - The kinetic scheme
   - A numerical experiment

The air entrainment

- appears in the transient flow in closed pipes not completely filled : the liquid flow (as well as the air flow) is free surface.




(a) Settings

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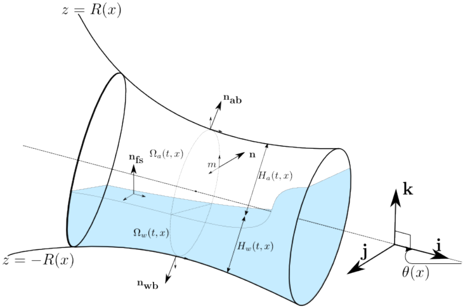
- may lead to two-phase flows for transition : free surface flows → pressurized flows.




(b) Settings

(c) Forced pipe

The air entrainment

- appears in the transient flow in closed pipes not completely filled : the liquid flow (as well as the air flow) is free surface.
- may lead to two-phase flows for transition : free surface flows → pressurized flows.
- may cause severe damage due to the pressure surge.


(d) . . . at Minnesota http://www.sewerhistory.org/grfx/misc/disaster.htm

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# Previous works

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- the PFS equations (Ersoy *et al*, IJFV 2009, JSC 2011).
- the two layer model (Saint-Venant like) (Ersoy *et al* M2AN 2013).
- the "two layer" model (Euler) with artificial pressure (Ersoy *et al* Int. J. of CFD 2015, 2016).

- Almost all two-fluids models introduce several mathematical and numerical difficulties such as
  - the ill-posedness (Stewart and B. Wendroff, JCP, 84)
  - the presence of discontinuous fluxes
  - interface tracking (diffusion problem) $\rightarrow$ high order numerical methods are often required
  - preserving contact discontinuities
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- $\Rightarrow$ Kelvin-Helmholtz instability, for which the two-layer model is not a priori suitable

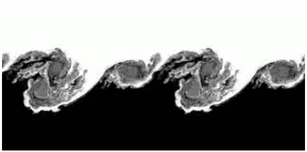

FIGURE: Kelvin-Helmholtz instability

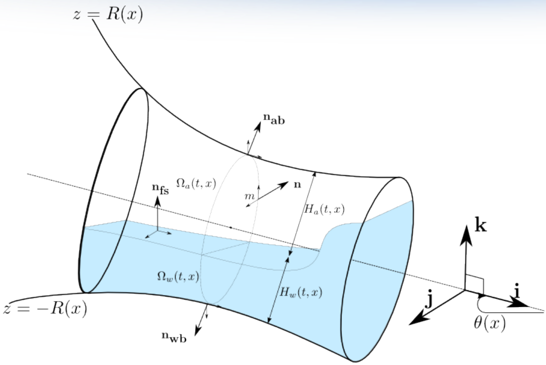


FIGURE: Geometric characteristics of the domain.

We have then the first natural coupling :

$$H_w(t,x) + H_a(t,x) = 2R(x)\,.$$

Outline

## Incompressible Euler's equations (Ersoy, Appl. of Mathematics, 2016)

$$\begin{array}{lcll} \mathrm{div}(\rho_0 \mathbf{U_w}) & = & 0, & \text{on } \mathbb{R}\times\Omega_{t,w} \\ \partial_t(\rho_0 \mathbf{U_w}) + \mathrm{div}(\rho_0 \mathbf{U_w}\otimes \mathbf{U_w}) + \nabla P_w & = & \rho_0 \mathbf{F}, & \text{on } \mathbb{R}\times\Omega_{t,w} \end{array}$$

where $\mathbf{U_w}(t,x,y,z) = (U_w, V_w, W_w)$ the velocity, $P_w(t,x,y,z)$ the pressure, $\mathbf{F}$ the gravity strength.

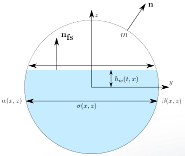


Figure: Cross-section of the domain

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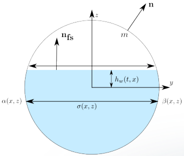
- Write non dimensional form of Euler equations using the parameter $\epsilon = H/L \ll 1$ and takes $\epsilon = 0$.
- Equality of the pressure of air and water $P_a = P_w$ at the free surface interface.
- Section averaging $\overline{\rho U} \approx \overline{\rho}\overline{U}$ and $\overline{U^2} \approx \overline{U}\,\overline{U}$.
- Introduce $A(t,x) = \int_{\Omega_w} dydz$, $u(t,x) = \frac{1}{A(t,x)} \int_{\Omega_w} U_w(t,x,y,z)\,dydz$, and $Q(t,x) = A(t,x)u(t,x)$.

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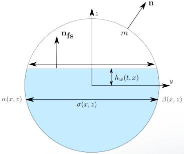


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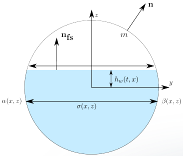


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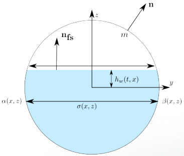


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## FLUID LAYER MODEL

$$\left\{\begin{array}{lcl} \partial_t A + \partial_x Q & = & 0 \\ \partial_t Q + \partial_x \left( \dfrac{Q^2}{A} + A P_a(\overline{\rho})/\rho_0 + g I_1(x,A)\cos\theta \right) & = & -gA\partial_x Z \\ & & +g I_2(x,A)\cos\theta \\ & & +P_a(\overline{\rho})/\rho_0\,\partial_x A \end{array}\right.$$

where

the hydrostatic pressure : $I_1(x,A) = \displaystyle\int_{-R}^{h_w} (h_w - z)\sigma(x,z)\,dz,$

the pressure source term : $I_2(x,A) = \displaystyle\int_{-R}^{h_w} (h_w - z)\partial_x\sigma(x,z)\,dz\,,$

the air pressure : $P_a$.

Outline

## Compressible Euler's equations (Ersoy, Asymptotic Analysis, 2016)

$$
\begin{aligned}
\partial_t \rho_a + \operatorname{div}(\rho_a \mathbf{U_a}) &= 0, \quad \text{on } \mathbb{R} \times \Omega_{t,a} \\
\partial_t (\rho_a \mathbf{U_a}) + \operatorname{div}(\rho_a \mathbf{U_a} \otimes \mathbf{U_a}) + \nabla P_a &= 0, \quad \text{on } \mathbb{R} \times \Omega_{t,a}
\end{aligned}
$$

where $\mathbf{U_a}(t,x,y,z) = (U_a, V_a, W_a)$ the velocity, $P_a(t,x,y,z)$ the pressure, $\rho_a(t,x,y,z)$ the density.

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with

$$P_a(\rho) = k\,\rho^\gamma \text{ with } k = \frac{p_a}{\rho_a^\gamma} \text{ where } \gamma \text{ is set to } 7/5.$$

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- Introduce $\mathcal{A}(t,x) = \int_{\Omega_a} dydz$, $v(t,x) = \frac{1}{\mathcal{A}(t,x)} \int_{\Omega_a} U_a(t,x,y,z)\,dydz$, $M = \overline{\rho}/\rho_0 \mathcal{A}$, $D = Mv$ and $c_a^2 = \frac{\partial p}{\partial \rho} = k\gamma \left(\frac{\rho_0 M}{\mathcal{A}}\right)^{\gamma-1}$.

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## AIR LAYER MODEL

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where

the $\gamma$ pressure : $\dfrac{M}{\gamma} c_a^2$,

the pressure source term : $\dfrac{M}{\gamma} c_a^2 \, \partial_x(\mathcal{A})$.

Outline

$\mathcal{A} + A = S$ where $S = S(x)$ denotes the pipe section

## Two-layer model

$$\left\{\begin{array}{rcl}
\partial_t M + \partial_x D & = & 0 \\
\partial_t D + \partial_x \left( \dfrac{D^2}{M} + \dfrac{M}{\gamma} c_a^2 \right) & = & \dfrac{M}{\gamma} c_a^2 \, \partial_x (S - A) \\
\partial_t A + \partial_x Q & = & 0 \\
\partial_t Q + \partial_x \left( \dfrac{Q^2}{A} + g I_1(x, A) \cos\theta + \dfrac{A}{(S - A)} \dfrac{M}{\gamma} c_a^2 \right) & = & -g A \partial_x Z \\
 & & + g I_2(x, A) \cos\theta \\
 & & + \dfrac{A}{(S - A)} \dfrac{M}{\gamma} c_a^2 \, \partial_x A
\end{array}\right. .$$

Outline

1. Energies

$$E_a = \frac{Mv^2}{2} + \frac{c_a^2\, M}{\gamma(\gamma-1)} \text{ and } E_w = \frac{Au^2}{2} + gA(h_w - I_1(x,A)/A)\cos\theta + gAZ$$

satisfy the following entropy flux equalities :

$$\partial_t E_a + \partial_x H_a = \frac{c_a^2\, M}{\gamma(S-A)}\,\partial_t A$$

and

$$\partial_t E_w + \partial_x H_w = -\,\frac{c_a^2\, M}{\gamma(S-A)}\,\partial_t A$$

where

$$H_a = \left(E_a + \frac{c_a^2\, M}{\gamma}\right) v \text{ and } H_w = \left(E_w + gI_1(x,A)\cos\theta + A\frac{c_a^2\, M}{(S-A)}\right) u\ .$$

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and

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where

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2. The total energy satisfies $\mathcal{E} = E_a + E_w$ the following equation

$$\partial_t \mathcal{E} + \partial_x \mathcal{H} = \boxed{0}\ .$$

- Quasi-linear form : $\mathbf{W} = (M, D, A, Q)^t$

$$\partial_t \mathbf{W} + \mathcal{D}(x, \mathbf{W}) \partial_X \mathbf{W} = 0$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ c_a^2 - v^2 & 2v & \dfrac{M}{S-A} c_a^2 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{A}{(S-A)} c_a^2 & 0 & c_w^2 + \dfrac{AM}{(S-A)^2} c_a^2 - u^2 & 2u \end{pmatrix}$$

where $c_m := c_w^2 + \dfrac{AM}{(S-A)^2} c_a^2$ : water sound speed under the air effect.

- Quasi-linear form : $\mathbf{W} = (M, D, A, Q)^t$

$$\partial_t \mathbf{W} + \mathcal{D}(x, \mathbf{W}) \partial_X \mathbf{W} = 0$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ c_a^2 - v^2 & 2v & \dfrac{M}{S-A} c_a^2 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{A}{(S-A)} c_a^2 & 0 & c_w^2 + \dfrac{AM}{(S-A)^2} c_a^2 - u^2 & 2u \end{pmatrix}$$

where $c_m := c_w^2 + \dfrac{AM}{(S-A)^2} c_a^2$ : water sound speed under the air effect.

- Writing

$$F = \frac{v-u}{c_m}, \quad \sqrt{H} = \frac{c_a}{c_m}, \quad c_m = \sqrt{c_w^2 + s c_a^2} \text{ with } s = \frac{AM}{(S-A)^2} = \frac{\rho}{\rho_0} \frac{A}{S-A} \geqslant 0,$$

the characteristic polynom reads $P(x = \lambda / c_m) =$

$$x^4 - 2(2+F)x^3 + ((1+F)(5+F) - H)\, x^2 + 2\left(H - (1+F)^2\right) x - sH^2$$

where $\lambda$ stands for an eigenvalue of $\mathcal{D}$.

## Theorem (Fuller, IEEE Trans. Automat. Control, 81)

*All the root of Equation* $P(x) = \sum_{k=0}^{4} a_k x^{4-k}$ *for* $(a_k)_k \in \mathbb{R}$ *and* $a_0 > 0$ *are real if and only if one of the following conditions holds :*

(*i*) $\Delta_3 > 0$, $\Delta_5 > 0$ *and* $\Delta_7 \geqslant 0$,

(*ii*) $\Delta_3 \geqslant 0$, $\Delta_5 = 0$ *and* $\Delta_7 = 0$

*where* $\Delta_3$, $\Delta_5$, $\Delta_7$ *are the inner determinant of the discriminant of* $P$.

$$\Delta_3 = \det \begin{pmatrix} a_0 & a_1 & a_2 \\ 0 & 4a_0 & 3a_1 \\ 4a_0 & 3a_1 & 2a_2 \end{pmatrix}, \quad \Delta_5 = \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \end{pmatrix},$$

$$\Delta_7 = \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \end{pmatrix}.$$

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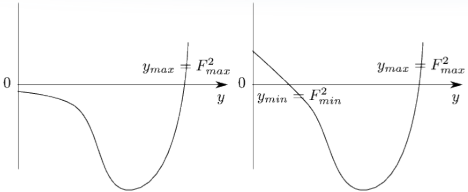
- From physical consideration, $\Delta_3 > 0$ and $\Delta_5 > 0 \Longrightarrow$ hyperbolic $\Longleftrightarrow \Delta_7 \geqslant 0$ where $\Delta_7(y = F^2) = 16HQ(y)$ with

$$\begin{aligned} Q(y) = \; & y^4 + \left(sH^2 + (s-4)H - 4\right) y^3 \\ & + \left((s^2 - 3s)H^3 + (6 - 26s)H^2 + (4 - 3s)H + 6\right) y^2 \\ & + \left((3s - 20s^2)H^4 + (13s - 20s^2 - 4)H^3 + (13s + 4)H^2 + (4s + 3)H - 4\right) y \\ & -(16s^3 + 8s^2 + s)H^5 + (32s^2 + 12s + 1)H^4 \\ & -(4 + 22s + 8s^2)H^3 + (12s + 6)H^2 - (4 + s)H + 1\,. \end{aligned}$$

- "conditionally" is due to $\Delta_7(y)$




(a) $y_{min}(H, s) < 0$ (b) $y_{min}(H, s) > 0$

FIGURE: Two real roots of the polynomial $\Delta_7(y)$

- Then,

## THEOREM (ERSOY *et al.*, M2AN,13)

*The two-layer system is conditionally hyperbolic and strictly hyperbolic if F satisfies one of the following conditions, for every $(H,s) \in \mathcal{D}$ or $\mathcal{D}^c$ (following if $R(H) \leqslant 0$ or $R(H) > 0$)*

- $y = F^2 \geqslant y_{max}(H,s) := F^2_{max} \iff$ *large relative speed*
- $y_{min}(H,s) > 0$ *and* $0 \leqslant F^2 \leqslant y_{min}(H,s) = F^2_{min} \iff$ *small relative speed*

*where*

$$\mathcal{D} = \left\{ (H,s) \in \mathbb{R}^2_+; \begin{array}{c} 0 < s < 4 \text{ and } l_3(s) < H < l_1(s), \\ \text{or } 4 < s < 6 \text{ and } l_1(s) < H < l_3(s), \\ \text{or } s > 6 \text{ and } H > l_1(s) \end{array} \right\}$$

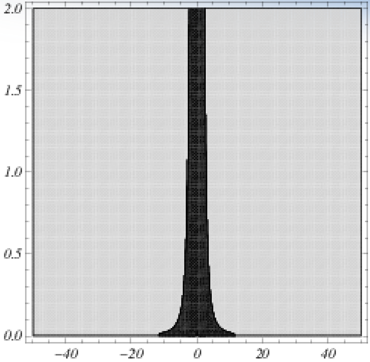
*with* $l_i(s)$, $i = 1, \ldots, 3$ *are the roots of the discriminant* $R(H) = 256H\left((s^2-6s)H^3 + (4s-12)H^2 + (40-6s)H - 12\right)$ *of*

$$\Delta_5(F;H,s) = 8\Big((1+H)F^4 - 2\left(H^2(1-s) + 1 - 6H\right)F^2 + (1+H)\left((H-1)^2 + 4sH^2\right)\Big),$$

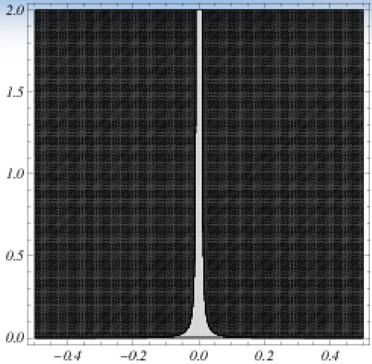
*i.e.*

$$l_1(s) = \frac{-1+\sqrt{1+6s}}{s}, \quad l_2(s) = -\frac{1+\sqrt{1+6s}}{s}, \quad l_3(s) = -\frac{2}{s-6}, \quad \text{and } l_4(s) = 0$$

## Hyperbolic region : examples $(s = s(\rho, A))$




(a) large relative speed ($\rho = 1000$ (air density), $F$ $x$-axis and $A$ $y$-axis)




(b) Small relative speed(Zoom on :$\rho = 1000$, $F$ $x$-axis and $A$ $y$-axis)

Figure: black grey = non hyperbolic region

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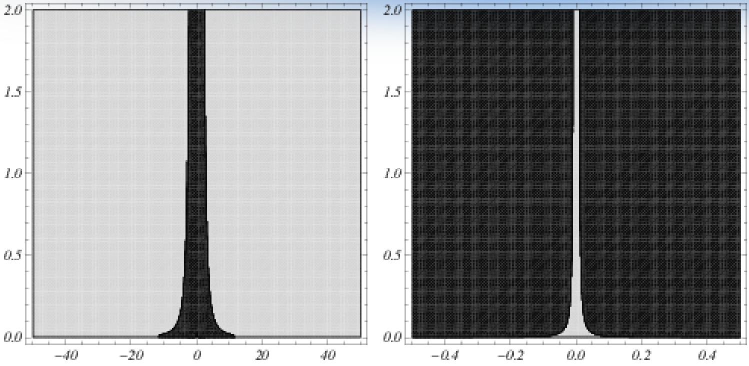


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- system may loses its hyperbolicity (range of validity).
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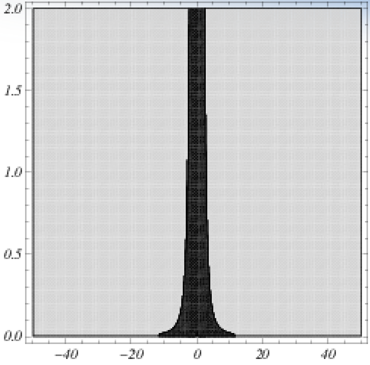


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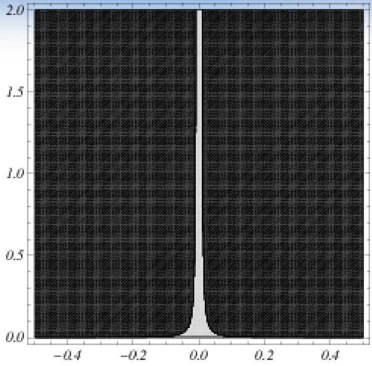


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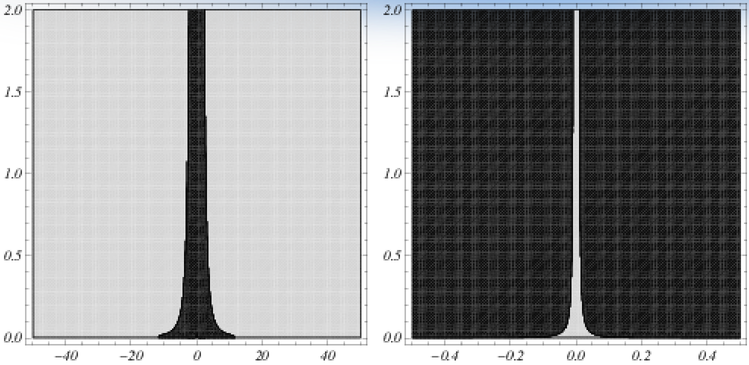
# HYPERBOLIC REGION : EXAMPLES $(s = s(\rho, A))$

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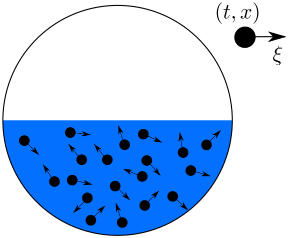
As in gas theory,
*Describe the macroscopic behavior from particle motions, here, assumed fictitious* by introducing
$\left\{\begin{array}{l}\text{a } \chi \text{ density function and} \\ \text{a } \mathcal{M}(t,x,\xi;\chi) \text{ maxwellian function (or a Gibbs equilibrium)}\end{array}\right.$



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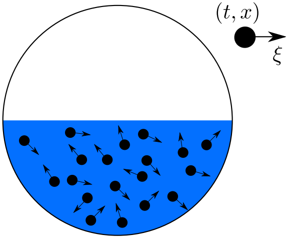


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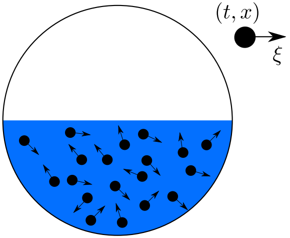


i.e., transform the nonlinear system into a kinetic transport equation on $\mathcal{M}$.
Thus, to be able to define the numerical *macroscopic fluxes* from the microscopic one.

*...Faire d'une pierre deux coups...*

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### MICRO-MACROSCOPIC RELATIONS

$$\begin{aligned} A &= \int_{\mathbb{R}} \mathcal{M}(t,x,\xi)\, d\xi \\ Au &= \int_{\mathbb{R}} \xi \mathcal{M}(t,x,\xi)\, d\xi \\ Au^2 + p &= \int_{\mathbb{R}} \xi^2 \mathcal{M}(t,x,\xi)\, d\xi \end{aligned}$$

## The kinetic formulation [Perthame, Oxford Lect. Ser. in Math. and its Applic., 02]

$(A, Q)$ is solution of the (air or water) system if and only if $\mathcal{M}$ satisfies the transport equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\Phi \, \partial_\xi \mathcal{M} = \mathcal{K}(t, x, \xi)$$

where $\mathcal{K}(t, x, \xi)$ is a collision kernel satisfying a.e. $(t, x)$

$$\int_{\mathbb{R}} \mathcal{K} \, d\xi = 0 \; , \; \int_{\mathbb{R}} \xi \, \mathcal{K} d \, \xi = 0$$

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General form of the source terms : $\Phi = \overbrace{\frac{d}{dx} Z}^{\text{conservative}} + \overbrace{\mathbf{B} \cdot \frac{d}{dx} \mathbf{W}}^{\text{non conservative}} + \overbrace{K \frac{Q|Q|}{A^2}}^{\text{friction}}$

- conservative term : classical upwind (Perthame, Simeoni, Calcolo 2001)
- non conservative term : mid point rule (Dal Maso, Lefloch, Murat, J. Math. Pures Appl., 95)
- friction : dynamic topography (Ersoy, Ph.D., 2010, Numerische Mathematik, 2014)

- Recalling that $Z$ is constant per cell

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- Then $\forall (t,x) \in [t_n, t_{n+1}[ \times \overset{\circ}{m}_i$

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$$\left\{ \begin{array}{lll} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}(t_n, x, \xi) \overset{def}{:=} \sqrt{b(t_n,x)}\chi\left(\dfrac{\xi - u(t_n,x)}{\sqrt{b(t_n,x)}}\right) \end{array} \right.$$

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$$\mathbf{U}_i^{n+1} = \begin{pmatrix} A_i^{n+1} \\ Q_i^{n+1} \end{pmatrix} \overset{def}{:=} \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f_i^{n+1}(\xi)\, d\xi$$

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$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x}\left(F^-_{i+1/2} - F^+_{i-1/2}\right) \text{ with } F^{\pm}_{i\pm\frac{1}{2}} = \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}^{\pm}_{i\pm\frac{1}{2}}(\xi)\, d\xi.$$

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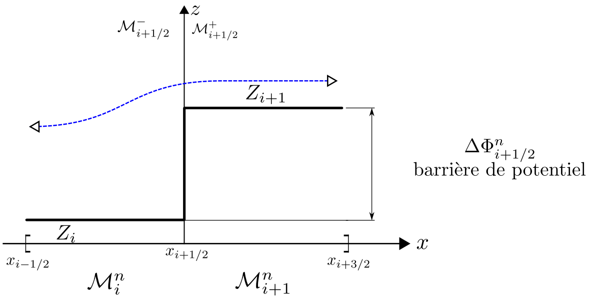


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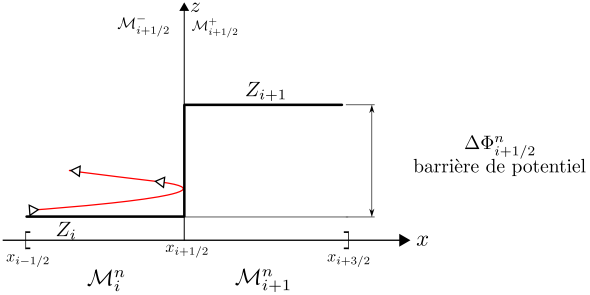


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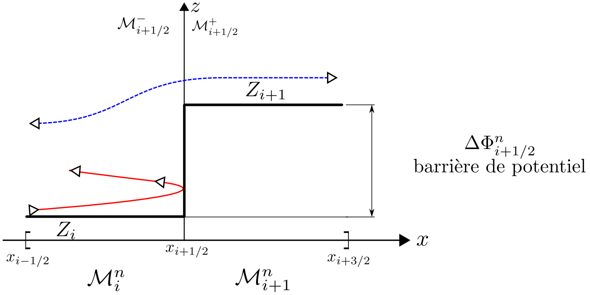


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## Properties [Ersoy et al., Numerische Mathematik, 2014]

Let $\chi$ be a compactly supported function, with $[-M, M]$ its support, verifying

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1 \ ,$$

Then,

1. The kinetic scheme is A-conservative.
2. Assume the following CFL condition $\Delta t^n \max_i \left( |u|_i^n + M \sqrt{\frac{p_i^n}{A_i^n}} \right) \leqslant \max_i h_i$ holds. Then, the kinetic scheme preserves the positivity of $A$.

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### Remark [Ersoy et al., Numerische Mathematik, 2014]

In practice we have used : $\chi = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3}, \sqrt{3}]}$

1. All integral terms are exact (implemented in the industrial code FlowMix (EDF, CIH, Chambéry).
2. Steady states are almost approximately preserved up to the order of the scheme.
3. Entropy inequalities are almost satisfied up to the order of the scheme.
4. A very good behavior when compared to experimental test cases.

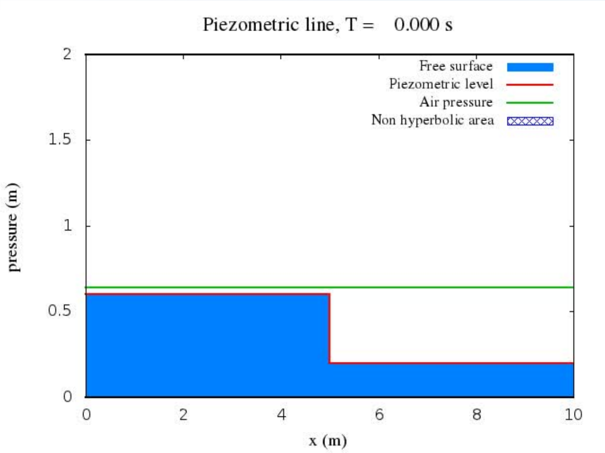


Figure: "Dam break in presence of air in a closed water pipe."

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  - apparition of vacuum, drying and flooding are obtained

## Conclusion

- Existence of a convex entropy function $\Rightarrow$ admissible weak solutions
- System is hyperbolic even for large relative speed
- Instability region
- Advantages of the kinetic scheme :
  - easy implementation
  - no use of eigenvalues $\Rightarrow$ computation in (non) hyperbolic region
  - apparition of vacuum, drying and flooding are obtained
  - equilibrium states are well-approximated

## Conclusion

- Existence of a convex entropy function $\Rightarrow$ admissible weak solutions
- System is hyperbolic even for large relative speed
- Instability region
- Advantages of the kinetic scheme :

## In progress

- air entrapment and mixed flows
- more realistic models based on interface instability tracking

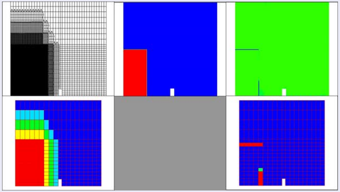

Figure: "Dam break in presence of obstacle." Ersoy et al, Cent. Europ. J. of Mathematics, 2013, Int. J. of CFD 2015,2016

# Thank you

# for your

# attention

# Upwinding of the source terms : $\Delta\Phi_{i+1/2}$

- conservative $\partial_x \mathbf{W}$ :

$$\mathbf{W}_{i+1} - \mathbf{W}_i$$

- non-conservative $\mathbf{B}\partial_x \mathbf{W}$ :

$$\overline{\mathbf{B}}(\mathbf{W}_{i+1} - \mathbf{W}_i)$$

where

$$\overline{\mathbf{B}} = \int_0^1 \mathbf{B}(s, \phi(s, \mathbf{W}_i, \mathbf{W}_{i+1}))\, ds$$

for the « straight lines paths », i.e.

$$\phi(s, \mathbf{W}_i, \mathbf{W}_{i+1}) = s\mathbf{W}_{i+1} + (1-s)\mathbf{W}_i,\ s \in [0,1]$$

*G. Dal Maso, P. G. Lefloch and F. Murat.*
Definition and weak stability of nonconservative products.
*J. Math. Pures Appl. , Vol 74(6) 483–548, 1995.*