Generalised Serre-Green-Naghdi equations for open channel and for natural river hydraulics

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- Modelling of open channel and rivers
  - water availability,
  - urban sewer systems,
  - flood risks,
  - ...

(a) Flooding

(b) DeltaFlume (NL)

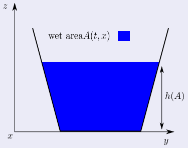
(c) Araguari River (Brazil)

- Esteves, Faucher, Galle, and Vauclin. Journal of hydrology, 2000.
- Torsvik, Pedersen, and Dysthe. Journal of waterway, port, coastal, and ocean engineering, 2009.

- Modelling of open channel and rivers
- Most widely used depth-averaged models :
  Saint-Venant system (hyperbolic, non linear, hydrostatic)

Depth averaged model

$$\left\{ \begin{array}{l} \partial_t h + \mathsf{div}(h\overline{u}) = 0, \\ \partial_t(h\overline{u}) + \mathsf{div}\left(h\overline{u}\otimes\overline{u} + g\dfrac{h^2}{2}I\right) = -gh\nabla d, \end{array}\right.$$



with

$h(t,x) = \eta(t,x) - d(x)$	:	water level
$\overline{u}(t,x) \in \mathbb{R}^2$	:	depth averaged speed
g	:	gravity

- Saint-Venant. Comptes rendus hebdomadaires des séances de l'Académie des sciences, 1871.
- Marche. Eur. J. Mech.B/ Fluids, 2007

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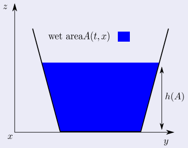
Section averaged model

$$\begin{cases} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(x, A) \right) = g I_2(x, A) \end{cases}$$

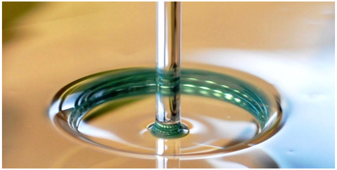


with

$A(t,x)$	:	wet area
$Q(t,x)$	:	discharge
$I_1(x,A) = \int_d^\eta \sigma(x,z)(\eta - z)dz$	:	hydrostatic pressure
$I_2(x,A) = \int_d^\eta \frac{\partial}{\partial x}\sigma(x,z)(\eta - z)dz$	:	hydrostatic pressure source
g	:	gravity

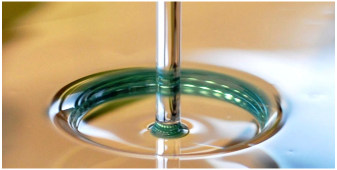
- Bourdarias, Ersoy, and Gerbi. Science China Mathematics, 2012.

- Modelling of open channel and rivers
- Most widely used depth-averaged models :
  Saint-Venant system (hyperbolic, non linear, hydrostatic)
- Hydrostatic models limitations → Illustration with undular bore
  - discontinuous solution also referred as bores takes the form of a breaking wave with turbulent rollers for large transitions.



(d) Bore

- Modelling of open channel and rivers
- Most widely used depth-averaged models :
  Saint-Venant system (hyperbolic, non linear, hydrostatic and non-dispersive)
- Hydrostatic models limitations → Illustration with undular bore
  - discontinuous solution also referred as bores takes the form of a breaking wave with turbulent rollers for large transitions.
  - the advancing front is followed by a train of free-surface undulations (whelps) for small or moderate transitions → dispersive effects



(f) Bore

(g) Undular bore

- Observation of Soliton

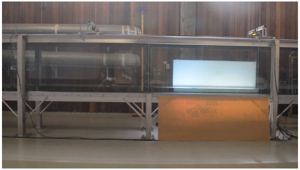


Figure – Russell's experiments "like" in 1834

- Observation of Soliton
- Dispersive equations (1D) introduced by Boussinesq in 1872 to justify mathematically the existence of solitary waves with $\varepsilon = O(\mu) \ll 1$

$$\left\{ \begin{array}{lcl} \frac{\partial}{\partial t}\xi + \frac{\partial}{\partial x}(h\overline{u}) & = & O(\mu^2) \\ \frac{\partial}{\partial t}\overline{u} + \varepsilon\overline{u}\frac{\partial}{\partial x}\overline{u} + \nabla\xi + \mu\mathcal{D} & = & O(\mu^2) \end{array} \right.$$

with

$\varepsilon = \frac{a}{H}$	:	non-linear parameter
$\mu = \left(\frac{H}{L}\right)^2$	:	dispersive parameter
h	:	water depth
ξ	:	free surface elevation
$\mathcal{D}$	:	dispersive term

Boussinesq. Comptes Rendus Acad. Sci, 1871.

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- Boussinesq. Comptes Rendus Acad. Sci, 1871.
- Korteweg and Gustav De Vries. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 1895.

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▶ Peregrine. Journal of fluid mechanics, 1967.

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- Witting. Journal of Computational Physics, 1984.
- Madsen and Sorensen. Coastal engineering, 1992.
- Nwogu. Journal of waterway, port, coastal, and ocean engineering, 1993.

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Serre. La Houille Blanche, 1953.

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- Recent progress : Lannes, Bonneton, Cienfuegos, Dutykh, Richard, Gavrilyuk, Sainte-Marie, . . .

State of the art & aims

Construction of a new averaged model for open channel and river flows considering that

- with 2D models $\rightarrow$ high memory and computer requirements.
- with 1D models $\rightarrow$ not accurate.

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- with 2D models $\rightarrow$ high memory and computer requirements.
- with 1D models $\rightarrow$ not accurate.
- good compromise can be achieved by 3D-1D model reduction
  - with non-linear terms
  - with dispersive terms
  - which takes into account of the channel/river geometry

Outline of the talk

1. Derivation (based on Euler equations)
   - 3D-2D
   - 2D-1D
   - 3D-1D

2. Improved model and stability
   - Reformulated and stable models
   - Invertible operator

3. Numerical analysis and test case
   - Finite Volume scheme
   - Numerical simulation

4. Conclusion and perspectives

OUTLINE

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   - 3D-2D
   - 2D-1D
   - 3D-1D
2. IMPROVED MODEL AND STABILITY
   - Reformulated and stable models
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4. CONCLUSION AND PERSPECTIVES

Incompressible and irrotational Euler equations

$$\begin{aligned} \operatorname{div}(\rho_0 \boldsymbol{u}) &= 0, \\ \frac{\partial}{\partial t}(\rho_0 \boldsymbol{u}) + \operatorname{div}(\rho_0 \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \rho_0 \boldsymbol{F} &= 0 \end{aligned}$$

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with

$\boldsymbol{u} = (u, v, w)$	:	velocity field
ρ_0	:	density
$\boldsymbol{F} = (0, 0, -g)$	:	external force
p	:	pressure

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completed with the irrotational relations

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} .$$

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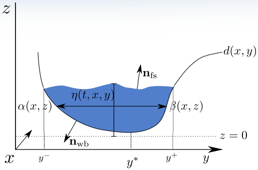

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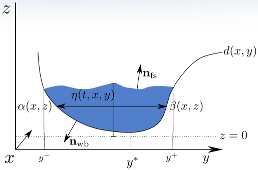

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- free surface kinematic boundary condition,

$$\boldsymbol{u} \cdot \boldsymbol{n}_{\mathrm{fs}} = \frac{\partial}{\partial t}\boldsymbol{m} \cdot \boldsymbol{n}_{\mathrm{fs}} \text{ and } p = p_0, \ \forall \boldsymbol{m}(t,x,y) = (x, y, \eta(t,x,y)) \in \Gamma_{\mathrm{fs}}(t,x)$$

- no-penetration condition on the wet boundary

$$\boldsymbol{u} \cdot \boldsymbol{n}_{\mathrm{wb}} = 0, \ \forall \boldsymbol{m}(x,y) = (x, y, d(x,y)) \in \Gamma_{\mathrm{wb}}(x)$$

Let us define the dispersive parameters

- $\mu_1 = \dfrac{h_1^2}{L^2}$
- $\mu_2 = \dfrac{H_2^2}{L^2}$,

such that

$$h_1 < H_1 = H_2 \ll L, \text{i.e. } \mu_1 < \mu_2^2$$

where

H_1	:	characteristic scale of channel width
h_1	:	characteristic wave-length in the transversal direction
H_2	:	characteristic water depth
$F_r = \dfrac{U}{\sqrt{gH_2}}$	:	Froude's number
$T = \dfrac{L}{U}$	:	characteristic time
$\mathcal{P} = U^2$	:	characteristic pressure
X	:	characteristic length of x

Then, define the dimensionless variables

$$\widetilde{x} = \frac{x}{L}, \quad \widetilde{P} = \frac{P}{\mathcal{P}}, \quad \widetilde{\varphi} = \frac{\varphi}{h_1},$$

$$\widetilde{y} = \frac{y}{h_1}, \quad \widetilde{u} = \frac{u}{U}, \quad \widetilde{d} = \frac{d}{H_2},$$

$$\widetilde{z} = \frac{z}{H_2}, \quad \widetilde{v} = \frac{v}{V} = \frac{v}{\sqrt{\mu_1}U}, \quad \widetilde{\eta} = \frac{\eta}{H_2} \ .$$

$$\widetilde{t} = \frac{t}{T}, \quad \widetilde{w} = \frac{w}{W} = \frac{w}{\sqrt{\mu_2}U} \ .$$

We get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} = 0$$

$$\mu_1\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial P}{\partial y} = 0$$

$$\mu_2\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \frac{\partial P}{\partial z} = -\frac{1}{{F_r}^2}$$

and

$$\frac{\partial u}{\partial y} = \mu_1\frac{\partial v}{\partial x},\ \ \mu_1\frac{\partial v}{\partial z} = \mu_2\frac{\partial w}{\partial y},\ \ \frac{\partial u}{\partial z} = \mu_2\frac{\partial w}{\partial x}\ .$$

"Coulisses" I : why $\mu_1 \neq \mu_2$?

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- $u_z = \mu w_x$

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  $w(t,x,z) = -\left(\int_d^z u_{|z=d}(t,x)\,dz\right)_x + O(\mu)$
- $\Rightarrow u(t,x,z) = f_1(u_{|z=d}(t,x)) + \mu f_2(z, u_{|z=d}(t,x), d(x)) + O(\mu^2) \Rightarrow$
  $u_{|z=d} = f_3(\bar{u}(t,x)) \ldots$

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Therefore, we assume $\mu_1 \neq \mu 2$.

A counter example if $h_1 = H_1$:

- Consider the (nondimensional) rectangular channel $(\tilde{x}, \tilde{y}, \tilde{z}) \in \left[0, \dfrac{L_c}{L}\right] \times \left[0, \dfrac{H_1}{h_1}\right] \times [0, 1]$ where $L \ll L_c$.

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- Incompressible + Irrotational $\Rightarrow \exists \tilde{\phi}\ ;\ (\tilde{u}, \tilde{v}, \tilde{w})^T = \nabla \tilde{\phi}$ solution of

$$\partial^2_{\tilde{x}\tilde{x}} \tilde{\phi} + \frac{1}{\mu_1} \partial^2_{\tilde{y}\tilde{y}} \tilde{\phi} + \frac{1}{\mu_2} \partial^2_{\tilde{z}\tilde{z}} \tilde{\phi} = 0\ .$$

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- Incompressible + Irrotational $\Rightarrow \exists \tilde{\phi}$; $(\tilde{u}, \tilde{v}, \tilde{w})^T = \nabla \tilde{\phi}$
- More precisely, $\forall (p, q) \in \mathbb{N}^2$, we have :

$$\tilde{\phi}_{p,q}(x, y, z) = \cos\left(p\pi \frac{\tilde{x}L}{L_c}\right) \cos\left(q\pi \frac{\tilde{y}h_1}{H_1}\right) \frac{\cosh\left(\pi \tilde{z} \sqrt{p^2 \mu_2 \frac{L^2}{L_c^2} + q^2 \frac{\mu_2}{\mu_1} \frac{h_1^2}{H_1^2}}\right)}{\cosh\left(\pi \sqrt{p^2 \mu_2 \frac{L^2}{L_c^2} + q^2 \frac{\mu_2}{\mu_1} \frac{h_1^2}{H_1^2}}\right)} .$$

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- Keeping in mind that $H_2 < L \ll L_c$,
  - if $h_1 = H_1 < H_2$ then

$$p^2 \mu_2 \frac{L^2}{L_c^2} + q^2 \frac{\mu_2}{\mu_1} \Rightarrow \tilde{u} = \partial_{\tilde{x}} \tilde{\phi} \text{ is rapidly varying in } \tilde{z}$$

  unless $H_1 > H_2$ (out of context)

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- Keeping in mind that $H_2 < L \ll L_c$,
  - if $h_1 = H_1 < H_2$ then is rapidly varying in $\tilde{z}$
  - Therefore, we consider $h_1 < H_1 = H_2$:

$$p^2\mu_2 \frac{L^2}{L_c^2} + q^2 \frac{\mu_2}{\mu_1}\frac{h_1^2}{H_1^2} = p^2 \frac{H_2^2}{L_c^2} + q^2 \frac{H_2^2}{H_1^2}$$

"Coulisses" III : order of integration

- "Coulisses" II naturally yields to $V < W < U$ where $(U, V = \sqrt{\mu_1}U, W = \sqrt{\mu_2}U)$
- As a consequence, we proceed as follows
  - 3D-2D reduction (width averaging)
  - 2D-1D reduction (depth averaging)
  - 3D-1D reduction (section averaging)

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$$u(t,x,y,z) = \langle u\rangle(t,x,z) + O(\mu_1)$$

  - 2D-1D reduction (depth averaging)
  - 3D-1D reduction (section averaging)

- "Coulisses" II naturally yields to $V < W < U$ where $(U, V = \sqrt{\mu_1}U, W = \sqrt{\mu_2}U)$
- As a consequence, we proceed as follows
  - 3D-2D reduction (width averaging) :

$$u(t,x,y,z) = \langle u \rangle(t,x,z) + O(\mu_1)$$

  - 2D-1D reduction (depth averaging) :

$$\langle u \rangle(t,x,z) = \overline{u}(t,x) + \mu_2 f(\overline{u}(t,x), \Omega(t,x)) + O(\mu_2^2)$$

    where $\overline{u}(t,x)$ is the section-averaged velocity
  - 3D-1D reduction (section averaging)

- "Coulisses" II naturally yields to $V < W < U$ where $(U, V = \sqrt{\mu_1} U, W = \sqrt{\mu_2} U)$
- As a consequence, we proceed as follows
  - 3D-2D reduction (width averaging) :

$$u(t,x,y,z) = \langle u \rangle (t,x,z) + O(\mu_1)$$

  - 2D-1D reduction (depth averaging) :

$$\langle u \rangle (t,x,z) = \overline{u}(t,x) + \mu_2 f(\overline{u}(t,x), \Omega(t,x)) + O(\mu_2^2)$$

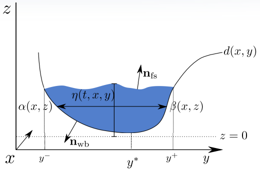
    where $\overline{u}(t,x)$ is the section-averaged velocity
  - 3D-1D reduction (section averaging) :

$$u(t,x,y,z) = \overline{u}(t,x) + \mu_2 f(\overline{u}(t,x), \Omega(t,x)) + O(\mu_2^2)$$

- Div and irrotational equations $\Rightarrow$

noting

$$X_\alpha(t,x,z) := X\left(t,x,\alpha(x,z),z\right)$$



we have

$$u(t,x,y,z) = u_\alpha(t,x,z) - \frac{\mu_1}{2}\frac{\partial}{\partial x}\mathrm{div}_{x,z}\left[\boldsymbol{w}_\alpha(t,x,z)(y-\alpha(x,z))^2\right] + O\left(\frac{\mu_1^2}{\mu_2}\right)$$

and

$$w(t,x,y,z) = w_\alpha(t,x,z) - \frac{\mu_1}{2\mu_2}\frac{\partial}{\partial z}\mathrm{div}_{x,z}\left[\boldsymbol{w}_\alpha(t,x,z)(y-\alpha(x,z))^2\right] + O\left(\frac{\mu_1^2}{\mu_2^2}\right)$$

Step 1 : 3D-2D reduction

- Width-averaging $\Rightarrow$ noting

$$\langle X\rangle(t,x,z) := \frac{1}{\sigma(x,z)}\int_{\alpha(x,z)}^{\beta(x,z)} X(t,x,y,z)\, dy$$

we have

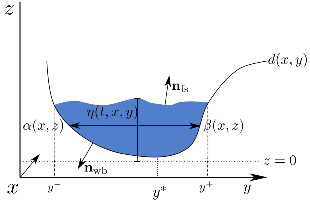
$$\sigma(x,z)\langle u\rangle(t,x,z) \quad = \quad \sigma(x,z)u_\alpha(t,x,z) - \frac{\mu_1}{6}\frac{\partial}{\partial x}\mathrm{div}_{x,z}\left[\boldsymbol{w}_\alpha(t,x,z)\sigma(x,z)^3\right] + O\left(\frac{\mu_1^2}{\mu_2}\right) ,$$

$$\sigma(x,z)\langle w\rangle(t,x,z) \quad = \quad \sigma(x,z)w_\alpha(t,x,z) - \frac{\mu_1}{6\mu_2}\frac{\partial}{\partial z}\mathrm{div}_{x,z}\left[\boldsymbol{w}_\alpha(t,x,z)\sigma(x,z)^3\right] + O\left(\frac{\mu_1^2}{\mu_2^2}\right)$$

where $\sigma(x,z) = \beta(x,z) - \alpha(x,z)$ is the width of the section at the elevation z.

- Width-averaging $\Rightarrow$

$$P(t,x,y,z) = P_\alpha(t,x,z) + O(\mu_1) = \frac{\eta(t,x,y)-z}{{F_r}^2} + \mu_2 \int_z^{\eta(t,x,y)} \frac{D}{Dt} w_\alpha(t,x,z)\, ds + O(\mu_1)$$



(a) Initial

a. Debyaoui, Ersoy, Asymptotic Analysis, 2020

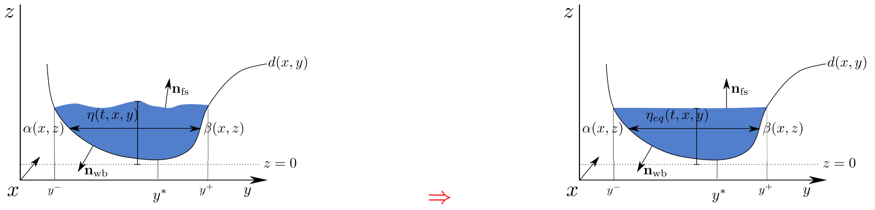
- Width-averaging ⇒

$$P(t,x,y,z) = P_\alpha(t,x,z) + O(\mu_1) = \frac{\eta(t,x,y)-z}{{F_r}^2} + \mu_2 \int_z^{\eta(t,x,y)} \frac{D}{Dt} w_\alpha(t,x,z)\, ds + O(\mu_1)$$

⇓

Flat free surface approximation [a] :

$$\eta(t,x,y) = \eta_{\mathrm{eq}}(t,x) + O(\mu_1)$$



(c) Initial

(d) Flat FS approximation

a. Debyaoui, Ersoy, Asymptotic Analysis, 2020

- Width-averaging $\Rightarrow$ we get the 2D width-averaged model

$$\begin{aligned}
\mathrm{div}_{x,z}\left[\sigma \boldsymbol{w}_\alpha\right] + O\left(\frac{\mu_1^2}{\mu_2^2}\right) &= \frac{\mu_1}{6\mu_2}\frac{\partial}{\partial z}\left(\sigma\frac{\partial}{\partial z}\left(\mathrm{div}_{x,z}\left[\boldsymbol{w}_\alpha \sigma^3\right]\right)\right) \\
\frac{\partial}{\partial t}(\sigma u_\alpha) + \mathrm{div}_{x,z}\left[\sigma u_\alpha \boldsymbol{w}_\alpha\right] + \frac{\partial}{\partial x}(\sigma P_\alpha) + O\left(\frac{\mu_1^2}{\mu_2^2}\right) &= P_\alpha\frac{\partial \sigma}{\partial x} \\
&\quad + \frac{\mu_1}{6\mu_2}\frac{\partial}{\partial x}\left(u_\alpha\frac{\partial}{\partial z}\mathrm{div}_{x,z}\left[\boldsymbol{w}_\alpha \sigma^3\right]\right) \\
\mu_2\left(\frac{\partial}{\partial t}(\sigma w_\alpha) + \mathrm{div}_{x,z}\left[\sigma w_\alpha \boldsymbol{w}_\alpha\right]\right) + \frac{\partial}{\partial z}(\sigma P_\alpha) &= -\frac{\sigma}{{F_r}^2} \\
&\quad + P_\alpha\frac{\partial \sigma}{\partial z} + O(\mu_1)
\end{aligned}$$

completed with the irrotational equation

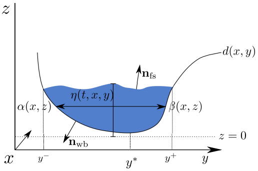
$$\frac{\partial u_\alpha}{\partial z} = \mu_2\frac{\partial w_\alpha}{\partial x} + O(\mu_1)$$

OUTLINE

1. DERIVATION (BASED ON EULER EQUATIONS)
   - 3D-2D
   - 2D-1D
   - 3D-1D
2. IMPROVED MODEL AND STABILITY
   - Reformulated and stable models
   - Invertible operator
3. NUMERICAL ANALYSIS AND TEST CASE
   - Finite Volume scheme
   - Numerical simulation
4. CONCLUSION AND PERSPECTIVES

- Div and irrotational equations (model 2D) $\Rightarrow$ noting

$$f_b(t,x) = f_\alpha(t,x,d^*(x)), \quad \mathcal{S}(u,x,z) = \frac{1}{\sigma(x,z)}\frac{\partial}{\partial x}\left(uS(x,z)\right), \quad S(x,z) = \int_{d^*(x)}^{z} \sigma(x,s)\, ds$$



we have

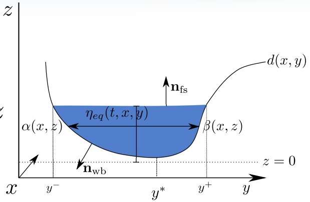
$$u_\alpha(t,x,z) = u_b(t,x) - \mu_2 \int_{d^*(x)}^{z} \frac{\partial}{\partial x}\mathcal{S}(u_b,x,s)\, ds + O(\mu_2^2)$$

and

$$w_\alpha(t,x,z) = -\frac{1}{\sigma(x,z)}\frac{\partial}{\partial x}\left(u_b(t,x)S(x,z)\right) + O(\mu_2)$$

- Depth-averaging $\Rightarrow$ noting

$$\bar{u}_{\text{eq}} = \frac{1}{A_{\text{eq}}(t,x)} \int_{d^*(x)}^{\eta_{\text{eq}}(t,x)} \int_{\alpha(x,z)}^{\beta(x,z)} u(t,x,y,z)\ dydz$$



we get

$$\begin{aligned} u_b(t,x) &= \bar{u}_{\text{eq}}(t,x) \\ &+ \frac{\mu_2}{A_{\text{eq}}(t,x)} \int_{d^*(x)}^{\eta_{\text{eq}}(t,x)} \sigma(x,z) \left(\int_{d^*(x)}^{z} \frac{\partial}{\partial x} \mathcal{S}(\bar{u}_{\text{eq}}(t,x), x, s)\ ds \right) dz \\ &+ O(\mu_2^2) \end{aligned}$$

- Depth-averaging $\Rightarrow$ finally,

$$u(t,x,y,z) = \bar{u}_{\text{eq}}(t,x) + \mu_2 B_0(\bar{u}_{\text{eq}},x,z) + O(\mu_2^2)$$

with

$$\begin{aligned} B_0(\bar{u}_{\text{eq}},x,z) &= \frac{1}{A_{\text{eq}}(t,x)} \int_{d^*(x)}^{\eta_{\text{eq}}(t,x)} \left(\sigma(x,z) \int_{d^*(x)}^{z} \frac{\partial}{\partial x} \mathcal{S}(\bar{u}_{\text{eq}}(t,x),x,s)\, ds \right) dz \\ &\quad - \int_{d^*(x)}^{z} \frac{\partial}{\partial x} \mathcal{S}(\bar{u}_{\text{eq}}(t,x),x,s)\, ds \end{aligned}$$

- Depth-averaging $\Rightarrow$ we also have

$$P(t,x,y,z) = P_{\mathrm{h}}(t,x,z) + \mu_2 P_{\mathrm{nh}}(t,x,z) + O(\mu_2^2)$$

where

$$P_{\mathrm{h}}(t,x,z) = \frac{(z - \eta_{\mathrm{eq}}(t,x))}{{F_r}^2}$$

and

$$\begin{aligned} P_{\mathrm{nh}}(t,x,z) &= \int_z^{\eta_{\mathrm{eq}}(t,x)} \frac{1}{2\sigma(x,s)^2}\frac{\partial}{\partial z}\left((\sigma(x,s)\mathcal{S}(\bar{u}_{\mathrm{eq}}(t,x),x,s))^2 \right)\, ds \\ &\quad - \int_z^{\eta_{\mathrm{eq}}(t,x)} \frac{\partial}{\partial t}\mathcal{S}(\bar{u}_{\mathrm{eq}}(t,x),x,s) \\ &\quad + \frac{\bar{u}_{\mathrm{eq}}(t,x)}{\sigma(x,s)}\frac{\partial}{\partial x}(\sigma(x,s)\mathcal{S}(\bar{u}_{\mathrm{eq}}(t,x),x,s))\, ds \end{aligned}$$

OUTLINE

- Euler equations in Ω_{eq} instead of Ω

- Euler equations in Ω_{eq} instead of Ω
- Boundary condition :

$$\int_{\partial\Omega_{\text{eq}}(t,x)} \left(\frac{\partial}{\partial t}\boldsymbol{M} + u\frac{\partial}{\partial x}\boldsymbol{M} - \boldsymbol{v}\right)\cdot\boldsymbol{n}\; ds = 0$$

STEP 3 : 3D-1D REDUCTION

- Euler equations in Ω_{eq} instead of Ω
- Boundary condition :

$$\int_{\partial\Omega_{\text{eq}}(t,x)} \left(\frac{\partial}{\partial t}\boldsymbol{M} + u\frac{\partial}{\partial x}\boldsymbol{M} - \boldsymbol{v}\right) \cdot \boldsymbol{n}\; ds = 0$$

- Introduce wet region indicator function Φ which satisfies

$$\frac{\partial}{\partial t}\Phi + \frac{\partial}{\partial x}(\Phi u) + \mathrm{div}_{y,z}\left[\Phi \boldsymbol{v}\right] = 0 \text{ on } \Omega_{\text{eq}}(t) = \bigcup_{0\leq x\leq 1} \Omega_{\text{eq}}(t,x) \; .$$

where $\boldsymbol{v} = (v, w)$.

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- Boundary condition :

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where $\boldsymbol{v} = (v, w)$.

- Section-averaging equations using the approximation

$$\begin{array}{lcl} u(t,x,y,z) & = & \bar{u}_{\mathrm{eq}}(t,x) + \mu_2 B_0(\bar{u}_{\mathrm{eq}}, x, z) + O(\mu_2^2) \\ \eta(t,x,y) & = & \eta_{\mathrm{eq}}(t,x) + O(\mu_1) \\ P(t,x,y,z) & = & P_{\mathrm{h}}(t,x,z) + \mu_2 P_{\mathrm{nh}}(t,x,z) + O(\mu_2^2) \end{array}$$

$$\left\{\begin{array}{l}\frac{\partial}{\partial t}A_{\mathrm{eq}}+\frac{\partial}{\partial x}Q_{\mathrm{eq}}=0\\ \frac{\partial}{\partial t}Q_{\mathrm{eq}}+\frac{\partial}{\partial x}\left(\frac{{Q_{\mathrm{eq}}}^{2}}{A_{\mathrm{eq}}}+I_1(x,A_{\mathrm{eq}})\right)+\mu_2\frac{\partial}{\partial x}(DI_1(x,A_{\mathrm{eq}},Q_{\mathrm{eq}}))=\\ I_2(x,A_{\mathrm{eq}})+\mu_2 DI_2(x,A_{\mathrm{eq}},Q_{\mathrm{eq}})+O(\mu_2^2)\end{array}\right.$$

where

$A_{\mathrm{eq}}=\int_{\Omega_{\mathrm{eq}}(t,x)} dy\,dz$: wet area

$Q_{\mathrm{eq}}=A_{\mathrm{eq}}(t,x)\bar{u}_{\mathrm{eq}}(t,x)$: discharge

- Debyaoui, Ersoy. Asymptotic Analysis, 2020

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}A_{\mathrm{eq}} + \frac{\partial}{\partial x}Q_{\mathrm{eq}} = 0 \\ \frac{\partial}{\partial t}Q_{\mathrm{eq}} + \frac{\partial}{\partial x}\left(\frac{{Q_{\mathrm{eq}}}^2}{A_{\mathrm{eq}}} + I_1(x, A_{\mathrm{eq}})\right) + \mu_2 \frac{\partial}{\partial x}(DI_1(x, A_{\mathrm{eq}}, Q_{\mathrm{eq}})) = \\ I_2(x, A_{\mathrm{eq}}) + \mu_2 DI_2(x, A_{\mathrm{eq}}, Q_{\mathrm{eq}}) + O(\mu_2^2) \end{array}\right.$$

where

$$I_1 = \int_{\Omega_{\mathrm{eq}}(t,x)} \frac{\eta_{\mathrm{eq}}(t,x) - z}{F_r^2}\sigma(x,z)\,dy\,dz \quad : \quad \text{hydro. press.}$$

$$I_2 = -\int_{y^-(t,x)}^{y^+(t,x)} \frac{h_{\mathrm{eq}}(t,x)}{{F_r}^2}\frac{\partial}{\partial x}d(x,y)\,dy \quad : \quad \text{hydro. press. source}$$

- Debyaoui, Ersoy. Asymptotic Analysis, 2020

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}A_{\text{eq}} + \dfrac{\partial}{\partial x}Q_{\text{eq}} = 0 \\ \dfrac{\partial}{\partial t}Q_{\text{eq}} + \dfrac{\partial}{\partial x}\left(\dfrac{{Q_{\text{eq}}}^2}{A_{\text{eq}}} + I_1(x, A_{\text{eq}})\right) + \mu_2 \dfrac{\partial}{\partial x}(DI_1(x, A_{\text{eq}}, Q_{\text{eq}})) = \\ I_2(x, A_{\text{eq}}) + \mu_2 DI_2(x, A_{\text{eq}}, Q_{\text{eq}}) + O(\mu_2^2) \end{array}\right.$$

where

$$DI_1 = \int_{\Omega_{\text{eq}}(t,x)} P_{\text{nh}}(t, x, z)\, dy\, dz$$: (disp) non hydro. press.

$$DI_2 = -\int_{y^-(t,x)}^{y^+(t,x)} P_{\text{nh}}(t, x, d(x, y)) \frac{\partial}{\partial x} d(x, y)\, dy$$: (disp) non hydro. press. source

- Debyaoui, Ersoy. Asymptotic Analysis, 2020

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} A_{\text{eq}} + \frac{\partial}{\partial x} Q_{\text{eq}} = 0 \\ \frac{\partial}{\partial t} Q_{\text{eq}} + \frac{\partial}{\partial x} \left(\frac{{Q_{\text{eq}}}^2}{A_{\text{eq}}} + I_1(x, A_{\text{eq}}) \right) + \mu_2 \frac{\partial}{\partial x} (DI_1(x, A_{\text{eq}}, Q_{\text{eq}})) = \\ I_2(x, A_{\text{eq}}) + \mu_2 DI_2(x, A_{\text{eq}}, Q_{\text{eq}}) + O(\mu_2^2) \end{array} \right.$$

REMARK (GENERALISATION OF THE FREE SURFACE MODEL)

Setting $\mu_2 = 0$, we recover the usual nlsw equations for open channel.

- Bourdarias, Ersoy, Gerbi. Science China Mathematics, 2012.
- Debyaoui, Ersoy. Asymptotic Analysis, 2020

Reformulation : generalization of the SGN equations

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial t}A_{\mathrm{eq}}+\dfrac{\partial}{\partial x}Q_{\mathrm{eq}}=0\\ \dfrac{\partial}{\partial t}Q_{\mathrm{eq}}+\dfrac{\partial}{\partial x}\left(\dfrac{{Q_{\mathrm{eq}}}^{2}}{A_{\mathrm{eq}}}+I_1(x,A_{\mathrm{eq}})\right)+\mu_2\dfrac{\partial}{\partial x}(\mathcal{D}(\bar{u}_{\mathrm{eq}})G(A_{\mathrm{eq}},x))=I_2(x,A_{\mathrm{eq}})\\ +\mu_2\mathcal{G}(\bar{u}_{\mathrm{eq}},S,\sigma)+O(\mu_2^2)\end{array}\right.$$

$$\left\{\begin{array}{l}\frac{\partial}{\partial t}A_{\mathrm{eq}}+\frac{\partial}{\partial x}Q_{\mathrm{eq}}=0\\ \frac{\partial}{\partial t}Q_{\mathrm{eq}}+\frac{\partial}{\partial x}\left(\frac{{Q_{\mathrm{eq}}}^2}{A_{\mathrm{eq}}}+I_1(x,A_{\mathrm{eq}})\right)+\mu_2\frac{\partial}{\partial x}\left(\mathcal{D}(\bar{u}_{\mathrm{eq}})G(A_{\mathrm{eq}},x)\right)=I_2(x,A_{\mathrm{eq}})\\ +\mu_2\mathcal{G}(\bar{u}_{\mathrm{eq}},S,\sigma)+O(\mu_2^2)\end{array}\right.$$

where

$$\mathcal{D}(\bar{u}_{\mathrm{eq}})=\left(\frac{\partial}{\partial x}\bar{u}_{\mathrm{eq}}\right)^2-\frac{\partial}{\partial t}\frac{\partial}{\partial x}\bar{u}_{\mathrm{eq}}-\bar{u}_{\mathrm{eq}}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\bar{u}_{\mathrm{eq}}$$

and

$$G(A_{\mathrm{eq}},x)=\int_{d^*(x)}^{\eta_{\mathrm{eq}}}\sigma(x,z)\int_z^{\eta_{\mathrm{eq}}}\frac{S(x,s)}{\sigma(x,s)}\,ds\,dz$$

$$\left\{\begin{array}{l}\frac{\partial}{\partial t}A_{\mathrm{eq}}+\frac{\partial}{\partial x}Q_{\mathrm{eq}}=0\\ \frac{\partial}{\partial t}Q_{\mathrm{eq}}+\frac{\partial}{\partial x}\left(\frac{{Q_{\mathrm{eq}}}^2}{A_{\mathrm{eq}}}+I_1(x,A_{\mathrm{eq}})\right)+\mu_2\frac{\partial}{\partial x}(\mathcal{D}(\bar{u}_{\mathrm{eq}})G(A_{\mathrm{eq}},x))=I_2(x,A_{\mathrm{eq}})\\ +\mu_2\mathcal{G}(\bar{u}_{\mathrm{eq}},S,\sigma)+O(\mu_2^2)\end{array}\right.$$

where

$$\begin{aligned}\mathcal{G}(u,S,\sigma) &= \int_z^{\eta_{\mathrm{eq}}}\frac{u^2}{\sigma(x,s)}\left(\frac{\frac{\partial}{\partial x}S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)}-\frac{\partial}{\partial x}\frac{\partial}{\partial x}S(x,s)\right)\\ &\quad+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\frac{S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)^2}\\ &\quad-\left(\frac{\partial}{\partial t}\bar{u}_{\mathrm{eq}}+\bar{u}_{\mathrm{eq}}\frac{\partial}{\partial x}\bar{u}_{\mathrm{eq}}\right)\frac{\frac{\partial}{\partial x}S(x,s)}{\sigma(x,s)}\,ds\end{aligned}.$$

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial t}A_{\mathrm{eq}}+\dfrac{\partial}{\partial x}Q_{\mathrm{eq}}=0\\\dfrac{\partial}{\partial t}Q_{\mathrm{eq}}+\dfrac{\partial}{\partial x}\left(\dfrac{{Q_{\mathrm{eq}}}^2}{A_{\mathrm{eq}}}+I_1(x,A_{\mathrm{eq}})\right)+\mu_2\dfrac{\partial}{\partial x}(\mathcal{D}(\bar{u}_{\mathrm{eq}})G(A_{\mathrm{eq}},x))=I_2(x,A_{\mathrm{eq}})\\+\mu_2\mathcal{G}(\bar{u}_{\mathrm{eq}},S,\sigma)+O(\mu_2^2)\end{array}\right.$$

Setting $\sigma=1$, $d=1$,

- $A_{\mathrm{eq}}=h_{\mathrm{eq}}$
- $S(x,z)\equiv S(z)\Rightarrow\mathcal{G}=0$ and $I_2=0$
- $G=\dfrac{{h_{\mathrm{eq}}}^3}{3}$
- $I_1=\dfrac{{h_{\mathrm{eq}}}^2}{2F_r^2}$

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}A_{\mathrm{eq}} + \dfrac{\partial}{\partial x}Q_{\mathrm{eq}} = 0 \\ \dfrac{\partial}{\partial t}Q_{\mathrm{eq}} + \dfrac{\partial}{\partial x}\left(\dfrac{{Q_{\mathrm{eq}}}^2}{A_{\mathrm{eq}}} + I_1(x, A_{\mathrm{eq}})\right) + \mu_2\dfrac{\partial}{\partial x}(\mathcal{D}(\bar{u}_{\mathrm{eq}})G(A_{\mathrm{eq}}, x)) = I_2(x, A_{\mathrm{eq}}) \\ +\mu_2\mathcal{G}(\bar{u}_{\mathrm{eq}}, S, \sigma) + O(\mu_2^2) \end{array}\right.$$

we recover the classical SGN equations on flat bottom

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}h_{\mathrm{eq}} + \dfrac{\partial}{\partial x}(h_{\mathrm{eq}}u_{\mathrm{eq}}) = 0 \\ \dfrac{\partial}{\partial t}(h_{\mathrm{eq}}u_{\mathrm{eq}}) + \dfrac{\partial}{\partial x}\left(h_{\mathrm{eq}}{u_{\mathrm{eq}}}^2 + \dfrac{{h_{\mathrm{eq}}}^2}{2F_r^2}\right) + \mu_2\dfrac{\partial}{\partial x}\left(\dfrac{{h_{\mathrm{eq}}}^3}{3}\mathcal{D}(u_{\mathrm{eq}})\right) = O(\mu_2^2) \end{array}\right.$$

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial t}A_{\text{eq}}+\dfrac{\partial}{\partial x}Q_{\text{eq}}=0\\ \dfrac{\partial}{\partial t}Q_{\text{eq}}+\dfrac{\partial}{\partial x}\left(\dfrac{{Q_{\text{eq}}}^2}{A_{\text{eq}}}+I_1(x,A_{\text{eq}})\right)+\mu_2\dfrac{\partial}{\partial x}(\mathcal{D}(\bar{u}_{\text{eq}})G(A_{\text{eq}},x))=I_2(x,A_{\text{eq}})\\ +\mu_2\mathcal{G}(\bar{u}_{\text{eq}},S,\sigma)+O(\mu_2^2)\end{array}\right.$$

Remark

Dispersive equation are usually characterized by third order term $\Rightarrow$ may create high frequencies instabilities

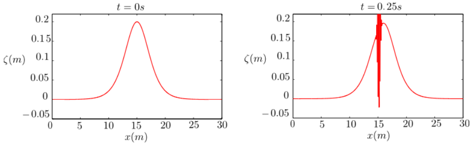

Figure – Bourdarias, Gerbi, and Ralph Lteif. Computers & Fluids, 156 :283–304, 2017.

OUTLINE

1. DERIVATION (BASED ON EULER EQUATIONS)
   - 3D-2D
   - 2D-1D
   - 3D-1D
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Outline

Outline

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators

$$\mathcal{T}[A_{\mathrm{eq}}, d, \sigma, z](u) = \frac{\partial}{\partial x}(u) \int_z^{\eta_{\mathrm{eq}}} \frac{S(x,s)}{\sigma(x,s)}\, ds + u \int_z^{\eta_{\mathrm{eq}}} \frac{1}{\sigma(x,s)} \frac{\partial}{\partial x} S(x,s)\, ds\ ,$$

and

$$\begin{aligned}\mathcal{G}[A_{\mathrm{eq}}, d, \sigma, z](u) \quad = \quad & \int_z^{\eta_{\mathrm{eq}}} 2\left(\frac{\partial}{\partial x} u\right)^2 \frac{S(x,s)}{\sigma(x,s)} + \\ & \frac{u^2}{\sigma(x,s)} \left(\frac{\frac{\partial}{\partial x} S(x,s) \frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)} - \frac{\partial}{\partial x}\frac{\partial}{\partial x} S(x,s) \right) \\ & + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \frac{S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)^2}\, ds\end{aligned}$$

A MORE STABLE FORMULATION→ USEFUL FOR NUMERICAL PURPOSE

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
- Define the averaged linear $\overline{\mathcal{T}}$ and the quadratic $\overline{\mathcal{Q}}$ operators

$$\overline{\mathcal{T}}[A_{\rm eq}, d, \sigma](u, \psi) = \int_{d^*(x)}^{\eta_{\rm eq}} \psi \mathcal{T}[A_{\rm eq}, d, \sigma, z](u) \, dz$$

and

$$\overline{\mathcal{G}}[A_{\rm eq}, d, \sigma](u, \psi) = \int_{d^*(x)}^{\eta_{\rm eq}} \psi \mathcal{G}[A_{\rm eq}, d, \sigma, z](u) \, dz$$

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
- Define the averaged linear $\overline{\mathcal{T}}$ and the quadratic $\overline{\mathcal{Q}}$ operators
- Define the operators $\mathcal{L}$ and $\mathcal{Q}$

$$\mathbb{L}[A_{\mathrm{eq}}, d, \sigma](u) \quad = \quad A_{\mathrm{eq}}\mathcal{L}[A_{\mathrm{eq}}, d, \sigma]\left(\frac{u}{A_{\mathrm{eq}}}\right)$$

and

$$\mathcal{Q}[A_{\mathrm{eq}}, d, \sigma](u) \quad = \quad \frac{1}{A_{\mathrm{eq}}}\left[\frac{\partial}{\partial x}\left(\overline{\mathcal{G}}[A_{\mathrm{eq}}, d, \sigma]\,(u, \sigma)\right) - \overline{\mathcal{G}}[A_{\mathrm{eq}}, d, \sigma]\left(u, \frac{\partial}{\partial x}\sigma\right)\right]$$

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
- Define the averaged linear $\overline{\mathcal{T}}$ and the quadratic $\overline{\mathcal{Q}}$ operators
- Define the operators $\mathcal{L}$ and $\mathcal{Q}$
- and finally the operator $\mathbb{L}$

$$\mathbb{L}[A_{\rm eq}, d, \sigma](u) \quad = \quad A_{\rm eq}\mathcal{L}[A_{\rm eq}, d, \sigma]\left(\frac{u}{A_{\rm eq}}\right)$$

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- Reformulated model

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial t}A_{\mathrm{eq}}+\dfrac{\partial}{\partial x}(A_{\mathrm{eq}}u_{\mathrm{eq}})=0\\(I_d-\mu_2\mathbb{L}[A_{\mathrm{eq}},d,\sigma])\left(\dfrac{\partial}{\partial t}(A_{\mathrm{eq}}u_{\mathrm{eq}})+\dfrac{\partial}{\partial x}\left(A_{\mathrm{eq}}{u_{\mathrm{eq}}}^2\right)\right)+\dfrac{\partial}{\partial x}I_1(x,A_{\mathrm{eq}})\\+\mu_2A_{\mathrm{eq}}\mathcal{Q}[A_{\mathrm{eq}},d,\sigma](u_{\mathrm{eq}})=I_2(x,A_{\mathrm{eq}})+O(\mu_2^2)\end{array}\right.$$

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Remark

Inverting $I_d-\mu_2\mathbb{L}[A_{\mathrm{eq}},d,\sigma]$ $\Rightarrow$ no third order term $\Rightarrow$ more stable formulation

- Bonneton, Barthélemy, Chazel, Cienfuegos, Lannes, Marche, and Tissier. European Journal of Mechanics-B/Fluids, 2011
- Debyaoui, Ersoy. Part 2, preprint, 2020

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Remark

A consistent one-parameter family (up to order $O(\mu_2^2)$) can be introduced to improve the frequency dispersion.

- Bonneton, Barthélemy, Chazel, Cienfuegos, Lannes, Marche, and Tissier. European Journal of Mechanics-B/Fluids, 2011
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- Reformulated model

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}A_{\mathrm{eq}} + \frac{\partial}{\partial x}(A_{\mathrm{eq}}u_{\mathrm{eq}}) = 0 \\ (I_d - \mu_2\kappa\mathbb{L}[A_{\mathrm{eq}}, d, \sigma]) \left(\frac{\partial}{\partial t}(A_{\mathrm{eq}}u_{\mathrm{eq}}) + \frac{\partial}{\partial x}(A_{\mathrm{eq}}{u_{\mathrm{eq}}}^2) + \frac{\kappa-1}{\kappa}\left(\frac{\partial}{\partial x}I_1 - I_2\right)\right) \\ + \frac{1}{\kappa}\left(\frac{\partial}{\partial x}I_1 - I_2\right) + \mu_2 A_{\mathrm{eq}}\mathcal{Q}[A_{\mathrm{eq}}, d, \sigma](u_{\mathrm{eq}}) = O(\mu_2^2) \end{array}\right.$$

Remark

A consistent one-parameter $\kappa > 0$ family (up to order $O(\mu_2^2)$) can be introduced to improve the frequency dispersion.

- Bonneton, Barthélemy, Chazel, Cienfuegos, Lannes, Marche, and Tissier. European Journal of Mechanics-B/Fluids, 2011
- Debyaoui, Ersoy. Part 2, preprint, 2020

OUTLINE

Theorem

Let α,β and $d \in C_b^\infty$ and $A \in W^{1,\infty}(\mathbb{R})$ such that $\inf_{x\in\mathbb{R}} A \geq A_0 > 0$. Then the operator

$$\mathbb{T} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is well-defined, one-to-one and onto.

- Debyaoui, Ersoy. Part 2, preprint, 2019

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- Let $\mu_2 \in (0,1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$ the space $H^1(\mathbb{R})$ endowed with the norm

$$\| u \|_{\mu_2}^2 = \| u \|_2^2 + \mu_2 \| u_x \|_2^2$$

Theorem

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- Let $\mu_2 \in (0,1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$
- Define the bilinear form $a(u,v)$

$$a(u,v) = (A\mathbb{T}u, v) = (Au, v) +$$

$$\mu_2 \left(A \left(\frac{A}{\sqrt{3} u_x} - \frac{\sqrt{3}}{2} d_x u \right), \left(\frac{A}{\sqrt{3} v_x} - \frac{\sqrt{3}}{2} d_x v \right) \right) + (A d_x u, d_x v)$$

THEOREM

Let α,β and $d \in C_b^\infty$ and $A \in W^{1,\infty}(\mathbb{R})$ such that $\inf\limits_{x \in \mathbb{R}} A \geq A_0 > 0$. Then the operator

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- Let $\mu_2 \in (0,1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$
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- Lax-Milgram theorem

$$\exists!\ u \in H^1_{\mu_2}(\mathbb{R})\ ;\ a(u,v) = (f,v),\ \forall v \in H^1_{\mu_2}(\mathbb{R}),\ f \in L^2(\mathbb{R})$$

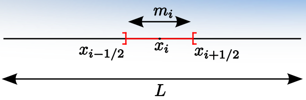
$$\Downarrow$$

$$\exists!\ u \in H^1_{\mu_2}(\mathbb{R})\ ;\ \mathbb{T}u = f$$

- From definition of $\mathbb{T}$, we get $u_{xx} = g(A,u,d,\sigma) \in L^2(\mathbb{R}) \Rightarrow u \in H^2(\mathbb{R})$.

Outline

OUTLINE



We consider a classical Finite Volume scheme, $\boldsymbol{U} = (A, Q)$

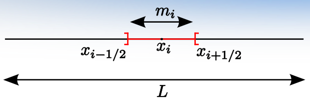
$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\delta t^n}{\delta x}\left(\boldsymbol{F}_{i+1/2}(\boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) - \boldsymbol{F}_{i-1/2}(\boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n)\right)$$

where $\boldsymbol{F}_{i\pm1/2} \approx \dfrac{1}{\delta t^n}\displaystyle\int_{m_i} \boldsymbol{F}(\boldsymbol{U}(t, x_{i+1/2}))\, dx$ is a Finite volume solver,

with

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} Au \\ Au^2 + \dfrac{\kappa - 1}{\kappa}\left(I_1 -'' \int I_2''\right) \end{pmatrix}$$

Numerical scheme : hyperbolic part



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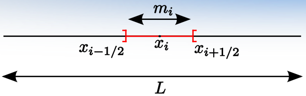
where $\boldsymbol{F}_{i\pm1/2} \approx \dfrac{1}{\delta t^n}\displaystyle\int_{m_i} \boldsymbol{F}(\boldsymbol{U}(t, x_{i+1/2}))\, dx$ is a Finite volume solver, for instance, with upwind technique to deal with source term

$$\boldsymbol{F}_{i\pm1/2} = \frac{\boldsymbol{F}(\boldsymbol{U}) + \boldsymbol{F}(\boldsymbol{V})}{2} - \frac{s_i^n}{2}(\boldsymbol{V} - \boldsymbol{U})$$

with

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} Au \\ Au^2 + \dfrac{\kappa - 1}{\kappa}\left(I_1 -'' \displaystyle\int I_2 ''\right)\end{pmatrix}$$

- Bourdarias, Ersoy, Gerbi. Journal of Scientific Computing, 2011



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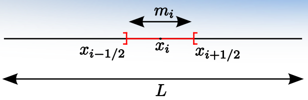
$$-\frac{\delta t^n}{\delta x}\left(\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1} \boldsymbol{D}^n\right)_i$$

with

$$(\boldsymbol{D}^n)_i = \boldsymbol{D}_{i+1/2}(\boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) - \boldsymbol{D}_{i-1/2}(\boldsymbol{U}_{i-2}^n, \boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n)$$

where $\boldsymbol{D}_{i\pm1/2}$ and $\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1}$ are the centred approximation of

$$\mathcal{D} = \frac{1}{\kappa}\left(\frac{\partial}{\partial x} I_1 - I_2\right) + \mu_2 A \mathcal{Q} \text{ and } \left[(I_d - \mu_2 \mathbb{L})\right]^{-1}$$



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$$-\frac{\delta t^n}{\delta x}\left(\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1} \boldsymbol{D}^n\right)_i$$

THEOREM

The numerical scheme is stable under the classical CFL condition,

$$\max_{\lambda \in \mathrm{Sp}(D_{\boldsymbol{U}} \boldsymbol{F}(\boldsymbol{U}))} |\lambda| \frac{\delta t^n}{\delta x} \leqslant 1 \ .$$

Debyaoui, Ersoy. NumHyp, 2020

Outline

Propagation of a solitary wave ($\kappa = 1$)

- Accuracy ($\sigma = d = 1$)

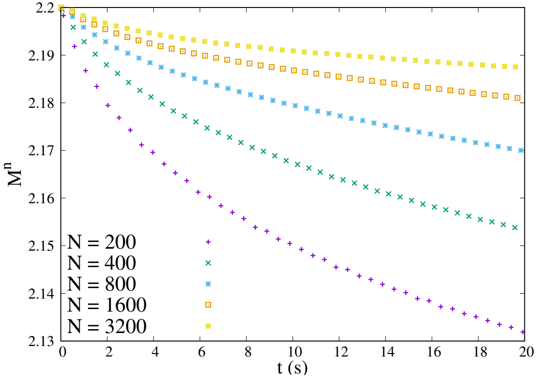

Figure – $M^n := \max_{0 \leq i \leq N+2} (h_i^n)$ and $M_{\text{soliton}}(t) := 2.2$

- Influence of the Section Variation ($N = 5000$ cells) :
  $\sigma(x;\varepsilon) = \beta(x;\varepsilon) - \alpha(x;\varepsilon)$ with
  $\beta = \frac{1}{2} - \frac{\varepsilon}{2}\exp\left(-\varepsilon^2\left(x - L/2\right)^2\right)$ and $\alpha = -\beta$

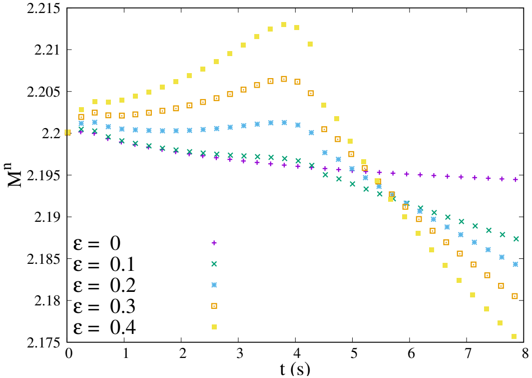

FIGURE – $M^n := \max_{x\in[0,L_c]} (h_i^n)$

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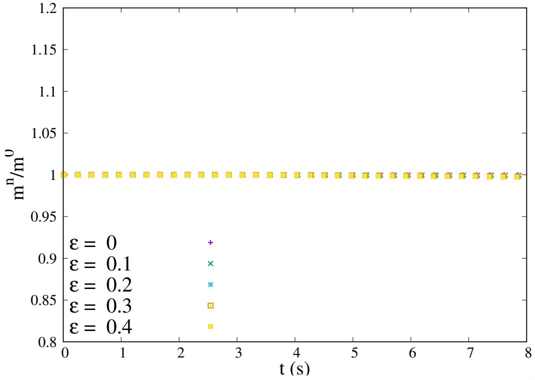


FIGURE – Influence of σ : $\dfrac{m^n}{m^0}$ with $m^n = \dfrac{1}{N+2}\sum_{i=0}^{N+1} A_i^n$

- Numerical order for $\varepsilon = 0$

N	$\Vert \eta\text{num} - \eta_{exact} \Vert_2$	$\Vert \eta\text{num} - \eta_{exact} \Vert_\infty$
100	0.0789	0.0449
200	0.0497	0.0288
400	0.0304	0.0180
800	0.0198	0.0116
1600	0.0153	0.0081
3200	0.0138	0.0062
Order	0.53	0.58

- **Numerical order** for $\varepsilon = 0.4$ (reference solution obtained with $N = 10000$ cells)

N	$\| \eta\text{num} - \eta_{\text{ref}} \|_2$	$\| \eta\text{num} - \eta_{\text{ref}} \|_\infty$
100	0.05212	0.02533
200	0.02096	0.01082
400	0.01079	0.00554
800	0.00748	0.00503
1600	0.00635	0.00412
3200	0.00505	0.00300
Order	0.64	0.56

- Comparison with the NLSW and the exact solution

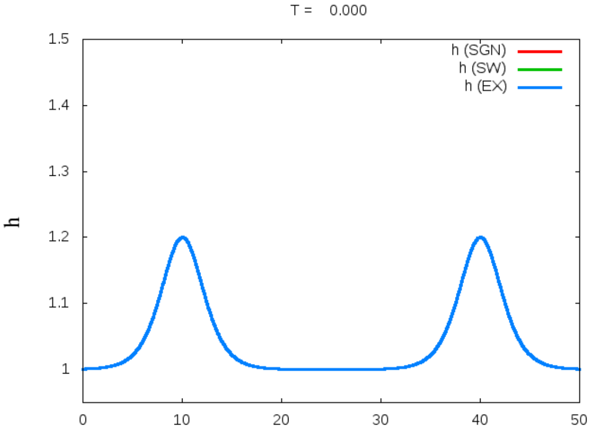
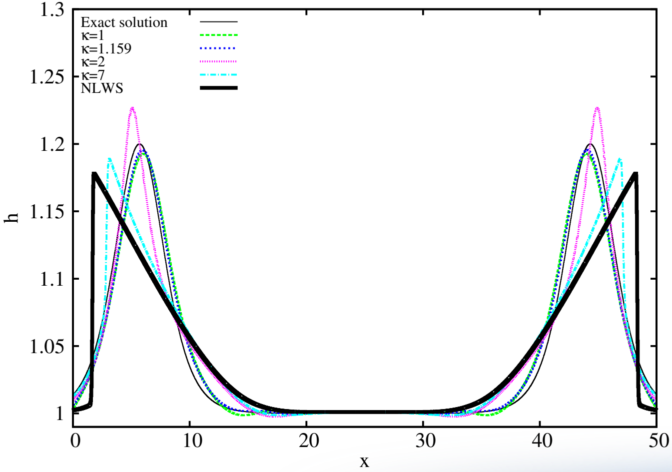


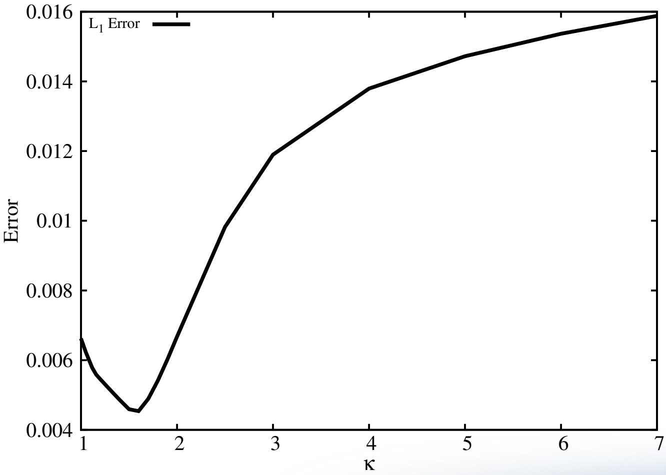
FIGURE – $\sigma = 1$, $d = 1$, $N = 1000$, $CFL = 0.95$, $T_f = 10$ and $\kappa = 1.159$

- Comparison with the NLSW and the exact solution
- Influence of κ



(b) Solutions at time $T_f = 10$

- Comparison with the NLSW and the exact solution
- Influence of κ



(d) $\| h_{ex} - h_{\kappa} \|_1$

OUTLINE

1. DERIVATION (BASED ON EULER EQUATIONS)
   - 3D-2D
   - 2D-1D
   - 3D-1D
2. IMPROVED MODEL AND STABILITY
   - Reformulated and stable models
   - Invertible operator
3. NUMERICAL ANALYSIS AND TEST CASE
   - Finite Volume scheme
   - Numerical simulation
4. CONCLUSION AND PERSPECTIVES

Conclusion

- Modeling
  - $\longrightarrow$ Non-linear
  - $\longrightarrow$ Dispersive
  - $\longrightarrow$ Non trivial geometry

Conclusion and perspectives

- Modeling
- Theoretical analysis

  $\longrightarrow$ Existence

  $\longrightarrow$ Special solutions

  $\longrightarrow$ Energy

Conclusion and perspectives

- Modeling
- Theoretical analysis
- Numerical analysis & Simulation
  $\longrightarrow$ Implementation of the general case

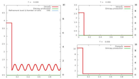
Conclusion and perspectives

- Modeling
- Theoretical analysis
- Numerical analysis & Simulation

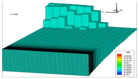
  ⟶ Implementation of the general case

  ⟶ Implementation in adaptive framework

Tools already developed for 1D, 2D and 3D problems



(k) 1D



(l) 2D

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Conclusion and perspectives

- Modeling
- Theoretical analysis
- Numerical analysis & Simulation

$\longrightarrow$ Implementation of the general case

$\longrightarrow$ Implementation in adaptive framework

$\longrightarrow$ Dissipative SGN (D-SGN) : switch from NLSW $\leftrightarrow$ SGN dynamically

CONCLUSION AND PERSPECTIVES

- Modeling
- Theoretical analysis
- Numerical analysis & Simulation

$\longrightarrow$ Implementation of the general case

$\longrightarrow$ Implementation in adaptive framework

$\longrightarrow$ Dissipative SGN (D-SGN) : switch from NLSW $\leftrightarrow$ SGN dynamically

$\longrightarrow$ 2D D-SGN – 1D D-SGN coupling

Thank you

for your

attention