A pressurized model for compressible pipe flows: derivation including friction.

M. Ersoy, IMATH, Toulon

MTM workshop

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Outline of the talk

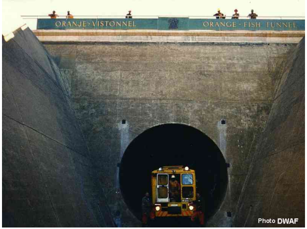
1. Physical background, Mathematical motivation and previous works
2. Derivation of the model including friction
3. Numerical experiment and concluding remarks

OUTLINE

1. PHYSICAL BACKGROUND, MATHEMATICAL MOTIVATION AND PREVIOUS WORKS
2. DERIVATION OF THE MODEL INCLUDING FRICTION
3. NUMERICAL EXPERIMENT AND CONCLUDING REMARKS

Simulation of pressurized flows

- plays an important role in many engineering applications such as
  - storm sewers
  - waste
  - or supply pipes in hydroelectric installations, ...



(a) Orange-Fish tunnel

(b) Sewers . . . in Paris

(c) Forced pipe

Pressurized flows : overview

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- "geyser" effect $\longrightarrow$ pressure can reach severe values and may cause irreversible damage !
- requiring efficient mathematical models and accurate numerical schemes

Pipe friction : definition and applications

- Friction law

$$F(u) = -k(u_\tau)u_\tau, \quad u_\tau : \text{tangential fluid flow}$$

- tangential constraint

$$\sigma(u)n \cdot \tau = \rho k(u_\tau)u_\tau, \quad \rho : \text{density}, \sigma : \text{total stress tensor}$$

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- Hydraulic engineering applications : canal, irrigation, dam-break, sediment transport, geyser, energy loss, failure pumping, fluid blockage, boundary layer, ...

How to determine the friction factor

The friction factor is called Fanning friction factor (whenever it is related to the shear stress) or Darcy friction factor (whenever it is related to the head loss $= 4\times$ Fanning friction factor) and

$$C = C(R_e, \delta, R_h, \ldots)\,.$$

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  - $C_l = \dfrac{C_0}{R_e}$
- transient flows
  - Colebrook (1939) formula : $\dfrac{1}{\sqrt{C}} = -2\log_{10}\left(\dfrac{\delta}{\alpha R_h} + \dfrac{\beta}{R_e\sqrt{C}}\right)$
  - or approximated Colebrook formula : Blasius, Haaland, Swamee-Jain,. . .

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- turbulent flows
  - Chézy (1776), Manning (1891), Strickler (1923) : $C_t = \dfrac{1}{K_s^2 R_h (S(x))^{4/3}}$

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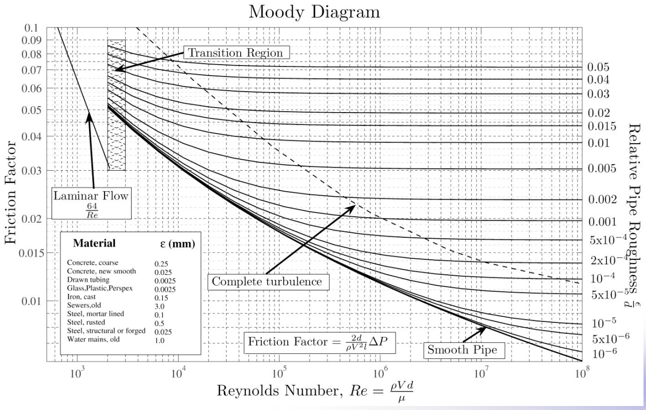
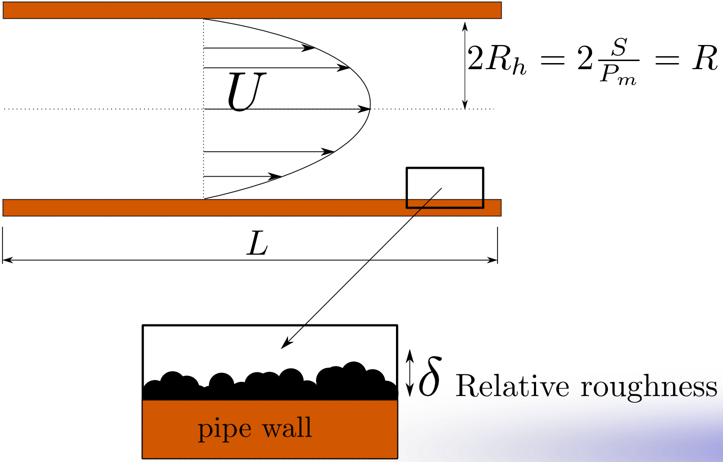
These coefficients are determined through the Moody diagram.



$$u = mean + oscillation$$
$$u = U + \widetilde{u}$$
$$R_e = \frac{\rho_0 U L}{\mu}$$



Mathematical motivations

- to reduce Viscous Compressible 3D NS $p(\rho) = c\rho \rightarrow$ inviscid compressible 1D SW-like model

J.-F. Gerbeau, B. Perthame
Derivation of viscous Saint-Venant system for laminar shallow water ; numerical validation.
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A model for unsteady mixed flows in non uniform closed water pipes : a Full Kinetic Approach.
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  - $u(t,x,y,z) = \overline{u(t,x)} + u(\widetilde{t,x,y},z)$,
  - $\displaystyle\int_\Omega u(\widetilde{t,x,y},z)\,dy\,dz = 0$,
  - $u(\widetilde{t,x,y},z) = O(\varepsilon)$ where ε is the aspect-ratio.
  - $\overline{u(t,x)^2} \approx \overline{u(t,x)}^2$

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- to include the friction with its geometrical dependency as well as other geometrical source terms
- general barotropic law $p(\rho) = c\rho^{\gamma}$, $\gamma \neq 1$
- $\overline{\rho^{\gamma}} \approx \overline{\rho}^{\gamma}$

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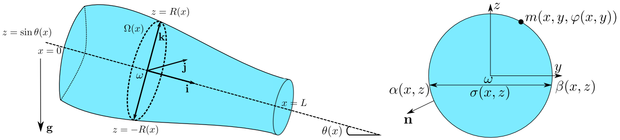
Let us consider a compressible fluid confined in a three dimensional domain $\mathcal{P}$, a non deformable pipe of length L oriented following the $\mathbf{i}$ vector,

$$\mathcal{P} := \left\{(x, y, z) \in \mathbb{R}^3;\ x \in [0, L],\ (y, z) \in \Omega(x)\right\}$$

where the section $\Omega(x)$, $x \in [0, L]$, is

$$\Omega(x) = \{(y, z) \in \mathbb{R}^2;\, y \in [\alpha(x, z), \beta(x, z)],\ z \in [-R(x), R(x)]\}$$



(d) Configuration (e) Ω-plane

Figure : Geometric characteristics of the pipe

The Compressible Navier-Stokes equations

$$\left\{ \begin{array}{rcl} \partial_t \rho + \mathsf{div}(\rho \mathbf{u}) & = & 0 \, , \\ \partial_t(\rho \mathbf{u}) + \mathsf{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mathsf{div}\sigma - \rho F & = & 0 \, , \\ p = p(\rho) = c\rho^\gamma \text{ with } \gamma & = & 1 \, , \end{array} \right.$$

velocity : $\mathbf{u} = \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix}$,

density : ρ,

gravity : $F = g \begin{pmatrix} \sin\theta(x) \\ 0 \\ -\cos\theta(x) \end{pmatrix}$,

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tensor : $\sigma = \begin{pmatrix} -p + \lambda\mathsf{div}(\mathbf{u}) + 2\mu\partial_x u & R(\mathbf{u})^t \\ R(\mathbf{u}) & -pI_2 + \lambda\mathsf{div}(\mathbf{u})I_2 + 2\mu D_{y,z}(\mathbf{v}) \end{pmatrix}$,

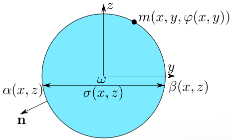
dynamical viscosity : μ,
volume viscosity : λ,

and $R(\mathbf{u}) = \mu\left(\nabla_{y,z} u + \partial_x \mathbf{v}\right)$, $\quad \nabla_{y,z} u = \begin{pmatrix} \partial_y u \\ \partial_z u \end{pmatrix}$, $\quad D_{y,z}(\mathbf{v}) = \nabla_{y,z}\mathbf{v} + \nabla^t_{y,z}\mathbf{v}$

Boundary conditions : inner wall $\partial\Omega(x)$, $\forall x \in (0, L)$

- wall-law condition including a general friction law k :

$$(\sigma(\mathbf{u})n_b) \cdot \tau_{b_i} = (\rho k(\mathbf{u})\mathbf{u}) \cdot \tau_{b_i}, \;\; x \in (0, L), \;\; (y, z) \in \partial\Omega(x)$$



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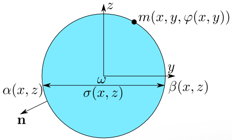
where τ_{b_i} is the i^{th} vector of the tangential basis. with

$$n_b = \frac{1}{\sqrt{(\partial_x \varphi)^2 + \mathbf{n} \cdot \mathbf{n}}} \begin{pmatrix} -\partial_x \varphi \\ \mathbf{n} \end{pmatrix} \text{ where } \mathbf{n} = \begin{pmatrix} -\partial_y \varphi \\ 1 \end{pmatrix}$$

is the outward normal vector in the Ω-plane.



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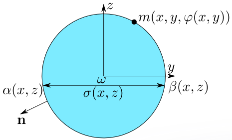
- completed with a no-penetration condition :

$$\mathbf{u}\cdot n_b = 0,\ \ x\in(0,L),\ (y,z)\in\partial\Omega(x)$$



THIN-LAYER ASSUMPTION AND ASYMPTOTIC ORDERING

- “thin-layer” assumption : $\varepsilon = \dfrac{D}{L} = \dfrac{W}{U} = \dfrac{V}{U} \ll 1$ and $T = \dfrac{L}{U}$

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- dimensionless quantities :
  - time $\tilde{t} = \dfrac{t}{T}$,
  - coordinate $(\tilde{x}, \tilde{y}, \tilde{z}) = \left(\dfrac{x}{L}, \dfrac{y}{D}, \dfrac{z}{D}\right)$
  - velocity field $(\tilde{u}, \tilde{v}, \tilde{w}) = \left(\dfrac{u}{U}, \dfrac{v}{W}, \dfrac{w}{W}\right)$
  - density $\tilde{\rho} = \dfrac{\rho}{\rho_0}$

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- dimensionless quantities : $\tilde{t}, (\tilde{x}, \tilde{y}, \tilde{z}), (\tilde{u}, \tilde{v}, \tilde{w}), \tilde{\rho}$
- non-dimensional numbers :

F_r	Froude number following the Ω-plane	:	$F_r = \dfrac{U}{\sqrt{gD}}$,
F_L	Froude number following the **i**-direction	:	$F_L = \dfrac{U}{\sqrt{gL}}$,
R_μ	Reynolds numbers with respect to μ	:	$R_\mu = \dfrac{\rho_0 U L}{\mu}$,
R_λ	Reynolds numbers with respect to λ	:	$R_\lambda = \dfrac{\rho_0 U L}{\lambda}$,
M_a	Mach number	:	$M_a = \dfrac{U}{c}$,
C	Oser number	:	$C = \dfrac{M_a}{F_r} = \dfrac{\sqrt{gD}}{c}$.

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- non-dimensional numbers : $F_r, F_L, R_\mu, R_\lambda, M_a, C$
- asymptotic ordering :

$$\boxed{R_\lambda^{-1} = \varepsilon\lambda_0, \quad R_\mu^{-1} = \varepsilon\mu_0, \quad K = \varepsilon K_0\ .}$$

The non-dimensional system

Dropping the $\tilde{\cdot}$, the system becomes :

$$
\begin{array}{rcl}
\partial_t \rho + \partial_x(\rho u) + \mathsf{div}_{y,z}(\rho \mathbf{v}) & = & 0 \ , \\
\partial_t(\rho u) + \partial_x(\rho u^2) + \mathsf{div}_{y,z}(\rho u \mathbf{v}) + \partial_x \dfrac{\rho}{M_a^2} & = & -\rho \dfrac{\sin\theta(x)}{F_L^2} + G_{\rho u} \\
 & & + \mathsf{div}_{y,z}\left(\dfrac{R_\mu^{-1}}{\varepsilon^2} \nabla_{y,z} u \right) \\
\varepsilon^2 \left(\partial_t(\rho \mathbf{v}) + \partial_x(\rho u \mathbf{v}) + \mathsf{div}_{y,z}(\rho \mathbf{v} \otimes \mathbf{v}) \right) + \nabla_{y,z} \dfrac{\rho}{M_a^2} & = & \begin{pmatrix} 0 \\ -\dfrac{\rho \cos\theta(x)}{F_r^2} \end{pmatrix} + G_{\rho \mathbf{v}} \ ,
\end{array}
$$

where the source terms are

$$
\begin{array}{rcl}
G_{\rho u} & = & \mathsf{div}_{y,z}\left(R_\mu^{-1} \partial_x \mathbf{v} \right) + \partial_x \left(2 R_\mu^{-1} \partial_x u + R_\lambda^{-1} \mathsf{div}(\mathbf{u}) \right) \ , \\
G_{\rho \mathbf{v}} & = & \partial_x \left(\varepsilon R_\varepsilon(\mathbf{u}) \right) + \mathsf{div}_{y,z}\left(R_\lambda^{-1} \mathsf{div}(\mathbf{u}) + 2 R_\mu^{-1} D_{y,z}(\mathbf{v}) \right) \ .
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R_\lambda^{-1} = \varepsilon \lambda_0, \quad R_\mu^{-1} = \varepsilon \mu_0
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The first order approximation

Formally, dropping all terms of order $O(\varepsilon)$, we obtain the so-called hydrostatic approximation :

$$\begin{array}{rcl}
\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) + \mathsf{div}_{y,z}(\rho_\varepsilon \boldsymbol{v}_\varepsilon) & = & 0 \\
\partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) + \mathsf{div}_{y,z}(\rho_\varepsilon u_\varepsilon \boldsymbol{v}_\varepsilon) + \dfrac{1}{M_a^2}\partial_x \rho_\varepsilon & = & -\rho_\varepsilon \dfrac{\sin\theta(x)}{F_L^2} \\
 & & +\mathsf{div}_{y,z}\left(\dfrac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon\right) \\
\dfrac{1}{M_a^2}\nabla_{y,z}\rho_\varepsilon & = & \begin{pmatrix} 0 \\ -\frac{\rho_\varepsilon \cos\theta(x)}{F_r^2} \end{pmatrix}
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Formally, dropping all terms of order $O(\varepsilon)$, we obtain the so-called hydrostatic approximation :

$$\begin{array}{rcl}
\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) + \mathsf{div}_{y,z}(\rho_\varepsilon \boldsymbol{v}_\varepsilon) & = & 0 \\
\partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) + \mathsf{div}_{y,z}(\rho_\varepsilon u_\varepsilon \boldsymbol{v}_\varepsilon) + \dfrac{1}{M_a^2}\partial_x \rho_\varepsilon & = & -\rho_\varepsilon \dfrac{\sin\theta(x)}{F_L^2} \\
 & & + \boxed{\mathsf{div}_{y,z}\left(\dfrac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon\right)} \\
\dfrac{1}{M_a^2}\nabla_{y,z}\rho_\varepsilon & = & \begin{pmatrix} 0 \\ -\frac{\rho_\varepsilon \cos\theta(x)}{F_r^2} \end{pmatrix}
\end{array}$$

REMARK

Let us emphasize that even if this system results from a formal limit of Equations as ε goes to 0, we note its solution $(\rho_\varepsilon, u_\varepsilon, \boldsymbol{v}_\varepsilon)$ due to the explicit dependency on ε.

The boundary conditions

- Boundary conditions : $\forall x \in (0, L),\ (y, z) \in \partial\Omega(x)$:

$$\frac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon \cdot \mathbf{n} = \rho_\varepsilon K_0(u) + O(\varepsilon) \quad \text{and} \quad \mu_0 \nabla_{y,z} u_\varepsilon = \boldsymbol{O}(\varepsilon) \ .$$

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- Momentum equation on $\rho_\varepsilon u_\varepsilon$:

$$\begin{aligned}\partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) + \mathsf{div}_{y,z}(\rho_\varepsilon u_\varepsilon \boldsymbol{v}_\varepsilon) + \frac{1}{M_a^2}\partial_x\rho_\varepsilon \quad &= \quad -\rho_\varepsilon\frac{\sin\theta(x)}{F_L^2} \\ &\quad +\mathsf{div}_{y,z}\left(\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\right)\end{aligned}$$

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$$\begin{aligned}\partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) + \mathrm{div}_{y,z}(\rho_\varepsilon u_\varepsilon \boldsymbol{v}_\varepsilon) + \frac{1}{M_a^2}\partial_x \rho_\varepsilon &= -\rho_\varepsilon \frac{\sin\theta(x)}{F_L^2} \\ &\quad + \mathrm{div}_{y,z}\left(\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\right)\end{aligned}$$

- Order $\frac{1}{\varepsilon}$:

$$\mathrm{div}_{y,z}\left(\mu_0 \nabla_{y,z}u_\varepsilon\right) = O(\varepsilon)$$

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- Order $\dfrac{1}{\varepsilon}$:

$$\mathrm{div}_{y,z}\left(\mu_0 \nabla_{y,z}u_\varepsilon\right) = O(\varepsilon)$$

- Neumann condition

$$\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon \cdot \mathbf{n} = \rho_\varepsilon K_0(u) + O(\varepsilon) \longrightarrow \mu_0 \nabla_{y,z}u_\varepsilon \cdot \mathbf{n} = O(\varepsilon)$$

The boundary conditions & the Neumann problem

- Boundary conditions : $\forall x \in (0,L),\ (y,z) \in \partial\Omega(x)$:

$$\frac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon \cdot \mathbf{n} = \rho_\varepsilon K_0(u) + O(\varepsilon) \quad \text{and} \quad \mu_0 \nabla_{y,z} u_\varepsilon = \boldsymbol{O}(\varepsilon)\ .$$

- Neumann problem

$$\left\{\begin{array}{lcll} \mathsf{div}_{y,z}\left(\mu_0 \nabla_{y,z} u_\varepsilon\right) & = & O(\varepsilon)\ , & (y,z) \in \Omega(x) \\ \mu_0 \partial_{\mathbf{n}} u_\varepsilon & = & O(\varepsilon)\ , & (y,z) \in \partial\Omega(x) \end{array}\right. .$$

- Boundary conditions : $\forall x \in (0, L),\ (y, z) \in \partial\Omega(x)$:

$$\frac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon \cdot \mathbf{n} = \rho_\varepsilon K_0(u) + O(\varepsilon) \quad \text{and} \quad \mu_0 \nabla_{y,z} u_\varepsilon = \boldsymbol{O}(\varepsilon) \ .$$

- Neumann problem

$$\left\{\begin{array}{lcl} \mathsf{div}_{y,z}\left(\mu_0 \nabla_{y,z} u_\varepsilon\right) & = & O(\varepsilon)\ , \quad (y,z) \in \Omega(x) \\ \mu_0 \partial_{\mathbf{n}} u_\varepsilon & = & O(\varepsilon)\ , \quad (y,z) \in \partial\Omega(x) \end{array}\right. .$$

$$\Downarrow$$

"motion by slices"

- "motion by slices"
  $u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x) + O(\varepsilon) \Longrightarrow u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x)$.

Consequence : first order approximation

- "motion by slices"
  $u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x) + O(\varepsilon) \Longrightarrow u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x)$.
- non-linearity : $\overline{u_\varepsilon^2} = \overline{u_\varepsilon}^2$.

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- non-linearity : $\overline{u_\varepsilon^2} = \overline{u_\varepsilon}^2$.
- stratified structure of the density :

$$\frac{1}{M_a^2}\nabla_{y,z}\rho_\varepsilon = \begin{pmatrix} 0 \\ -\dfrac{\rho_\varepsilon \cos\theta(x)}{F_r^2} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \partial_y \rho_\varepsilon \\ \partial_z \rho_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho_\varepsilon C^2 \cos\theta(x) \end{pmatrix}$$

$$\Downarrow$$

$\rho_\varepsilon(t,x,y,z) = \xi_\varepsilon(t,x)\exp\left(-C^2\cos\theta(x) z\right)$ for some positive function ξ_ε

Consequence : first order approximation

- "motion by slices"
  $u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x) + O(\varepsilon) \Longrightarrow u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x)$.
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$$\Downarrow$$

$$\rho_\varepsilon(t,x,y,z) = \xi_\varepsilon(t,x)\exp\left(-C^2\cos\theta(x) z\right) \text{ for some positive function } \xi_\varepsilon$$

$$\Downarrow$$

$$\overline{\rho_\varepsilon}(t,x) = \frac{\xi_\varepsilon(t,x)\Psi(x)}{S(x)}$$

$$\Psi(x) = \int_{\Omega(x)} \exp(-C^2\cos\theta(x)z)\, dy\, dz \quad : \quad \text{weighted pipe section} ,$$

$$S(x) = \int_{\Omega(t,x)} dydz \quad : \quad \text{physical pipe section} .$$

Consequence : first order approximation

- "motion by slices"
  $u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x) + O(\varepsilon) \Longrightarrow u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x)$.
- non-linearity : $\overline{u_\varepsilon^2} = \overline{u_\varepsilon}^2$.
- $\rho_\varepsilon(t,x,y,z) = \xi_\varepsilon(t,x)\exp\left(-C^2\cos\theta(x)z\right)$ for some positive function ξ_ε
- Momentum :

$$\overline{\rho_\varepsilon u_\varepsilon} = \frac{1}{S}\int_\Omega \rho_\varepsilon u_\varepsilon \; dydz = \frac{\xi_\varepsilon \Psi}{S}\overline{u_\varepsilon} = \overline{\rho_\varepsilon}\;\overline{u_\varepsilon}$$

Consequence : first order approximation

- "motion by slices"
  $u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x) + O(\varepsilon) \Longrightarrow u_\varepsilon(t,x,y,z) = \overline{u_\varepsilon}(t,x)$.
- non-linearity : $\overline{u_\varepsilon^2} = \overline{u_\varepsilon}^2$.
- $\rho_\varepsilon(t,x,y,z) = \xi_\varepsilon(t,x)\exp\left(-C^2\cos\theta(x)z\right)$ for some positive function ξ_ε
- $\overline{\rho_\varepsilon u_\varepsilon} = \overline{\rho_\varepsilon}\ \overline{u_\varepsilon}$
- $\overline{\rho_\varepsilon u_\varepsilon^2} = \overline{\rho_\varepsilon}\ \overline{u_\varepsilon^2}$

Consequence : first order approximation

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- $\rho_\varepsilon(t,x,y,z) = \xi_\varepsilon(t,x)\exp\left(-C^2\cos\theta(x)z\right)$ for some positive function ξ_ε
- $\overline{\rho_\varepsilon u_\varepsilon} = \overline{\rho_\varepsilon}\ \overline{u_\varepsilon}$
- $\overline{\rho_\varepsilon u_\varepsilon^2} = \overline{\rho_\varepsilon}\ \overline{u_\varepsilon^2} = \overline{\rho_\varepsilon}\ \overline{u_\varepsilon}^2$

The averaged model : P-model

- Integration of the hydrostatic equations over the cross-section Ω :

$$
\begin{aligned}
\partial_t(\overline{\rho_\varepsilon}S) + \partial_x(\overline{\rho_\varepsilon}S\overline{u_\varepsilon}) &= \int_{\partial\Omega(x)} \rho_\varepsilon\left(u_\varepsilon\partial_x\mathbf{m} - \boldsymbol{v}_\varepsilon\right)\cdot\mathbf{n}\,ds \\
\partial_t(\overline{\rho_\varepsilon}S\overline{u_\varepsilon}) + \partial_x\left(\overline{\rho_\varepsilon}S\overline{u_\varepsilon}^2 + \frac{1}{M_a^2}\overline{\rho_\varepsilon}S\right) &= -\overline{\rho_\varepsilon}S\frac{\sin\theta(x)}{F_L^2} + \frac{1}{M_a^2}\overline{\rho_\varepsilon}S\frac{d\,S}{dx} \\
&\quad + \int_{\partial\Omega(x)} \rho_\varepsilon u_\varepsilon\left(u_\varepsilon\partial_x\mathbf{m} - \mathbf{v}\right)\cdot\mathbf{n}\,ds \\
&\quad - \int_{\partial\Omega(x)} \frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\cdot\mathbf{n}\,ds
\end{aligned}
$$

  - Using Leibniz Formula
  - $\mathbf{m} = (y, \varphi(x,y)) \in \partial\Omega(x)$: the vector $\omega\mathbf{m}$
  - $\mathbf{n} = \dfrac{\mathbf{m}}{|\mathbf{m}|}$: the outward normal to $\partial\Omega(x)$ at $\mathbf{m}$ in the Ω-plane

The averaged model : P-model

- Integration of the hydrostatic equations over the cross-section Ω :

$$\begin{aligned}
\partial_t(\overline{\rho_\varepsilon} S) + \partial_x(\overline{\rho_\varepsilon} S \overline{u_\varepsilon}) &= \int_{\partial\Omega(x)} \rho_\varepsilon \left(u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon\right) \cdot \mathbf{n}\, ds \\
\partial_t(\overline{\rho_\varepsilon} S \overline{u_\varepsilon}) + \partial_x \left(\overline{\rho_\varepsilon} S \overline{u_\varepsilon}^2 + \frac{1}{M_a^2} \overline{\rho_\varepsilon} S\right) &= -\overline{\rho_\varepsilon} S \frac{\sin\theta(x)}{F_L^2} + \frac{1}{M_a^2} \overline{\rho_\varepsilon} S \frac{d\,S}{dx} \\
&\quad + \int_{\partial\Omega(x)} \rho_\varepsilon u_\varepsilon \left(u_\varepsilon \partial_x \mathbf{m} - \mathbf{v}\right) \cdot \mathbf{n}\, ds \\
&\quad - \int_{\partial\Omega(x)} \frac{\mu_0}{\varepsilon} \nabla_{y,z} u_\varepsilon \cdot \mathbf{n}\, ds
\end{aligned}$$

- no-penetration condition $\Longrightarrow$ $\left(u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon\right) \cdot \mathbf{n} = 0$

- Integration of the hydrostatic equations over the cross-section Ω :

$$\begin{array}{rcl}\partial_t(\overline{\rho_\varepsilon}S)+\partial_x(\overline{\rho_\varepsilon}S\overline{u_\varepsilon}) & = & 0\\ \partial_t(\overline{\rho_\varepsilon}S\overline{u_\varepsilon})+\partial_x\left(\overline{\rho_\varepsilon}S\overline{u_\varepsilon}^2+\dfrac{1}{M_a^2}\overline{\rho_\varepsilon}S\right) & = & -\overline{\rho_\varepsilon}S\dfrac{\sin\theta(x)}{F_L^2}+\dfrac{1}{M_a^2}\overline{\rho_\varepsilon}S\dfrac{d\,S}{dx}\\ & & -\displaystyle\int_{\partial\Omega(x)}\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\cdot\mathbf{n}\,ds\end{array}$$

- Friction term : $\displaystyle\int_{\partial\Omega(x)}\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\cdot\mathbf{n}\,ds=\int_{\partial\Omega(x)}\rho_\varepsilon K_0(u_\varepsilon)\;ds=$

$$\left(\frac{\xi_\varepsilon\Psi(x)}{S}\right)S\left(K_0(\overline{u_\varepsilon})\frac{\psi(x)}{\Psi(x)}\right)=\overline{\rho_\varepsilon}SK(x,\overline{u_\varepsilon})$$

ψ : the curvilinear integral of $z\to\exp(-C^2\cos\theta(x)z)$ along $\partial\Omega(x)$ called *weighted wet perimeter*.

The averaged model : P-model

- Integration of the hydrostatic equations over the cross-section Ω :

$$\begin{array}{rcl}
\partial_t(\overline{\rho_\varepsilon}S) + \partial_x(\overline{\rho_\varepsilon}S\overline{u_\varepsilon}) & = & 0 \\
\partial_t(\overline{\rho_\varepsilon}S\overline{u_\varepsilon}) + \partial_x\left(\overline{\rho_\varepsilon}S\overline{u_\varepsilon}^2 + \dfrac{1}{M_a^2}\overline{\rho_\varepsilon}S\right) & = & -\overline{\rho_\varepsilon}S\dfrac{\sin\theta(x)}{F_L^2} + \dfrac{1}{M_a^2}\overline{\rho_\varepsilon}S\dfrac{d\,S}{dx} \\
 & & -\overline{\rho_\varepsilon}SK\left(x, \overline{u_\varepsilon}\right)
\end{array}$$

- ψ : weighted wet perimeter of $\Omega \Longrightarrow \left(\dfrac{\psi(x)}{\Psi(x)}\right)^{-1}$: weighted hydraulic radius
  - Meaning that the friction is also a function of the Oser number
  - Neglected by engineers since $\psi =$ wet perimeter.

The averaged model : P-model

- Integration of the hydrostatic equations over the cross-section Ω :

$$
\begin{aligned}
\partial_t(\overline{\rho_\varepsilon} S) + \partial_x(\overline{\rho_\varepsilon} S \overline{u_\varepsilon}) &= 0 \\
\partial_t(\overline{\rho_\varepsilon} S \overline{u_\varepsilon}) + \partial_x\left(\overline{\rho_\varepsilon} S \overline{u_\varepsilon}^2 + c^2 \overline{\rho_\varepsilon} S\right) &= -g \overline{\rho_\varepsilon} S \sin\theta(x) + c^2 \overline{\rho_\varepsilon} S \frac{d\,S}{dx} \\
&\quad -g \overline{\rho_\varepsilon} S K\left(x, \overline{u_\varepsilon}\right)
\end{aligned}
$$

- multiply Equations by $\dfrac{\rho_0 D U^2}{L}$

- Integration of the hydrostatic equations over the cross-section Ω :

$$
\begin{array}{rcl}
\partial_t(A) + \partial_x(A\overline{u_\varepsilon}) & = & 0 \\
\partial_t(A\overline{u_\varepsilon}) + \partial_x\left(A\overline{u_\varepsilon}^2 + c^2 A\right) & = & -gA\sin\theta(x) + c^2\dfrac{A}{S}\dfrac{d\,S}{dx} \\
 & & -gAK\left(x, \overline{u_\varepsilon}\right)
\end{array}
$$

- multiply Equations by $\dfrac{\rho_0 D U^2}{L}$
- set $A = \overline{\rho_\varepsilon} S$: the wet area

The averaged model : P-model

- Integration of the hydrostatic equations over the cross-section Ω :

$$\begin{array}{rcl} \partial_t(A) + \partial_x(Q) & = & 0 \\ \partial_t(Q) + \partial_x\left(\dfrac{Q^2}{A} + c^2 A\right) & = & -gA\sin\theta(x) + c^2 \dfrac{A}{S}\dfrac{dS}{dx} \\ & & -gAK\left(x, \dfrac{Q}{A}\right) \end{array}$$

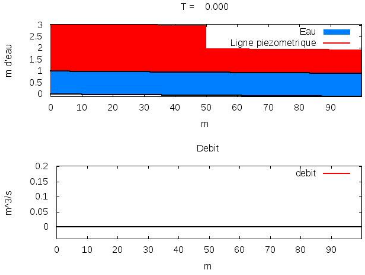
- multiply Equations by $\dfrac{\rho_0 D U^2}{L}$
- set $A = \overline{\rho_\varepsilon} S$: the wet area
- set $Q = A\overline{u_\varepsilon}$: the discharge

Outline

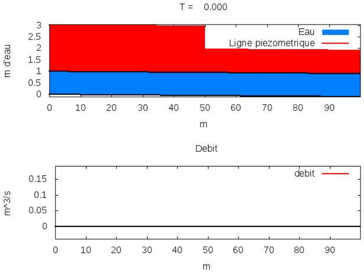
Outline

A "DAM-BREAK" LIKE EXPERIMENT (C=1)

- Generalized kinetic scheme introduced by Bourdarias, Ersoy and Gerbi (2014)
- Manning-Strickler friction law ($K_s = \dfrac{1}{M}$).
- We consider :
  Horizontal circular pipe : $L = 100\ m$, $D = 1\ m$.



(a) $M = 0$



(b) $M = 0.2$

A "dam-break" like experiment (C=1)

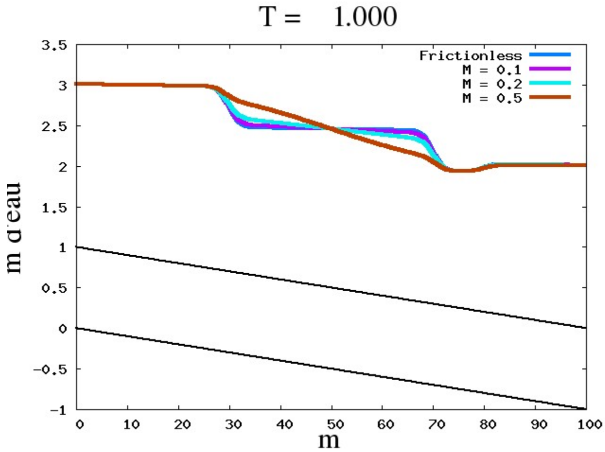
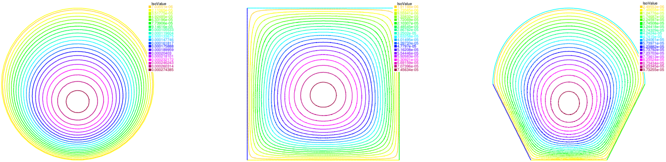


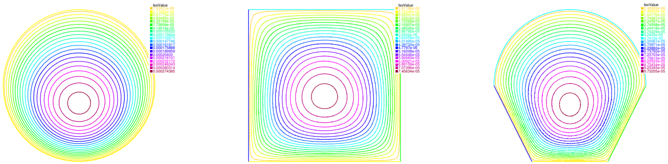
Figure : Influence of the friction

- the case $p(\rho) = \rho^{\gamma}$, $\gamma = 1$: second order approximation
  $(\varepsilon = 10^{-3}, C = 1) \longrightarrow$ paraboloid profile



Concluding remarks

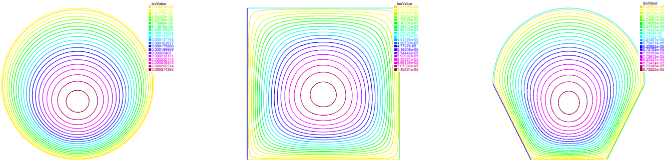
- the case $p(\rho) = \rho^{\gamma}$, $\gamma = 1$: second order approximation $(\varepsilon = 10^{-3}, C = 1) \longrightarrow$ paraboloid profile



- the case $p(\rho) = \rho^{\gamma}$, $\gamma \neq 1$

Concluding remarks

- the case $p(\rho) = \rho^\gamma$, $\gamma = 1$: second order approximation $(\varepsilon = 10^{-3}, C = 1) \longrightarrow$ paraboloid profile

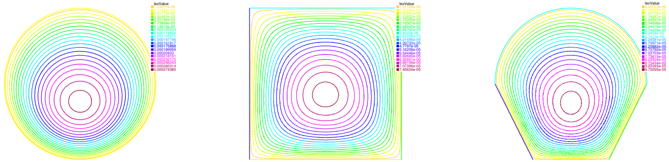


- the case $p(\rho) = \rho^\gamma$, $\gamma \neq 1$
  - hydrostatic equation $\longrightarrow \rho_\varepsilon(t,x,y,z) = \xi_\varepsilon(t,x) N(t,x,z)$ where

$$N(t,x,z) = \left(1 + zC^2 \cos\theta(x) \frac{1-\gamma}{\gamma \xi_\varepsilon(t,x)^{\gamma-1}}\right)^{\frac{1}{\gamma-1}}$$

Concluding remarks

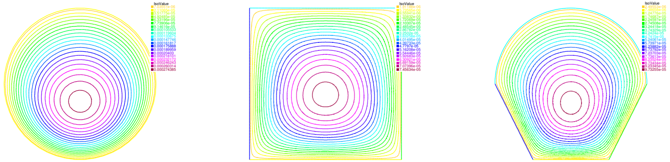
- the case $p(\rho) = \rho^\gamma$, $\gamma = 1$: second order approximation $(\varepsilon = 10^{-3}, C = 1) \longrightarrow$ paraboloid profile



- the case $p(\rho) = \rho^\gamma$, $\gamma \neq 1$
  - hydrostatic equation $\longrightarrow \rho_\varepsilon(t, x, y, z) = \xi_\varepsilon(t, x) N(t, x, z)$ where
    $$N(t, x, z) = \left(1 + zC^2 \cos\theta(x) \frac{1-\gamma}{\gamma \xi_\varepsilon(t, x)^{\gamma-1}}\right)^{\frac{1}{\gamma-1}}$$
  - the assumption $\overline{\rho^\gamma} \approx \overline{\rho}^\gamma$ is wrong ! ! !

Concluding remarks

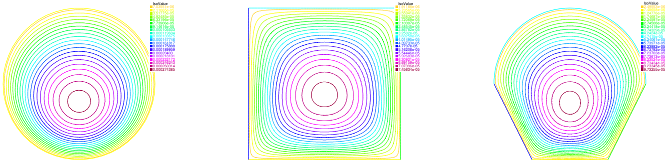
- the case $p(\rho) = \rho^\gamma$, $\gamma = 1$: second order approximation $(\varepsilon = 10^{-3}, C = 1) \longrightarrow$ paraboloid profile



- the case $p(\rho) = \rho^\gamma$, $\gamma \neq 1$
  - hydrostatic equation $\longrightarrow \rho_\varepsilon(t,x,y,z) = \xi_\varepsilon(t,x) N(t,x,z)$ where
  $$N(t,x,z) = \left(1 + zC^2 \cos\theta(x) \frac{1-\gamma}{\gamma \xi_\varepsilon(t,x)^{\gamma-1}}\right)^{\frac{1}{\gamma-1}}$$
  - the assumption $\overline{\rho^\gamma} \approx \overline{\rho}^\gamma$ is wrong ! ! !
  - except if the Oser number $C \ll 1 \longrightarrow$ a class of low Oser compressible γ models. This occurs when the gravity has no influence.

Concluding remarks

- the case $p(\rho)=\rho^\gamma$, $\gamma=1$: second order approximation $(\varepsilon=10^{-3}, C=1)\longrightarrow$ paraboloid profile

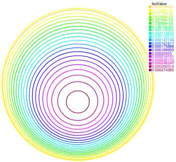
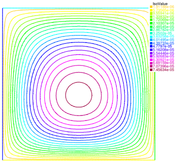
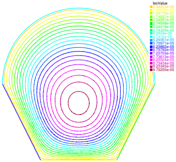


- the case $p(\rho)=\rho^\gamma$, $\gamma\neq 1$
  - First order Pressurized γ model can be derived in a similar way :

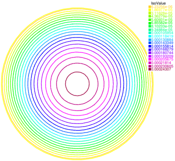
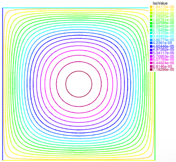
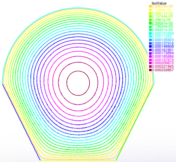
$$
\begin{array}{rcl}
\partial_t(\xi_\varepsilon S)+\partial_x(\xi_\varepsilon S\overline{u}) & = & 0 \\
\partial_t(\xi_\varepsilon S\overline{u_\varepsilon})+\partial_x\left(\xi_\varepsilon S\overline{u_\varepsilon}^2+\dfrac{1}{M_a^2}\xi_\varepsilon^\gamma S\right) & = & -\xi_\varepsilon S\dfrac{\sin\theta(x)}{F_L^2}+\dfrac{1}{M_a^2}\xi_\varepsilon^\gamma\dfrac{dS}{dx} \\
 & & -\xi_\varepsilon K(x,\overline{u_\varepsilon})
\end{array}
$$

Concluding remarks

- the case $p(\rho) = \rho^\gamma$, $\gamma = 1$: second order approximation $(\varepsilon = 10^{-3}, C = 1) \longrightarrow$ paraboloid profile

- the case $p(\rho) = \rho^\gamma$, $\gamma \neq 1$
  - Second order approximation $(\varepsilon = 10^{-3}, C = 10^{-3})$: paraboloid profile

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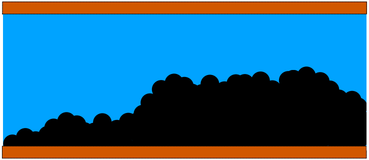

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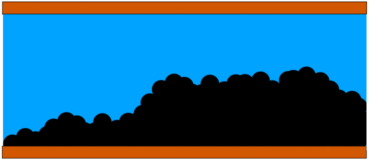
- to find
  - optimal pipe shape
  - including variable rugosity



(e) what happen inside the pipe

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Thank you

for your

attention