On a new mathematical model for open channel and river hydraulics

Mehmet Ersoy

2021, 16 November, ACSIOM, Montpellier, France

Outline

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- Introducing characteristic scales :
  - length L
  - width l
  - height H

- Introducing characteristic scales : L, l and H
- Introducing aspect ratio numbers :
  - $\varepsilon_z = \dfrac{H}{L}$ following the depth
  - $\varepsilon_y = \dfrac{l}{L}$ following the width

- Introducing characteristic scales : L, l and H
- Introducing aspect ratio numbers : $\varepsilon_z = \dfrac{H}{L}$ and $\varepsilon_y = \dfrac{l}{L}$
- One can reduce the initial model (Navier-Stokes or Euler equations)
  - 3D-2D depth averaged model reduction if

$$\varepsilon_z \ll 1 \text{ and } \varepsilon_y \approx 1$$

  - 3D-1D section averaged model reduction if

$$\varepsilon_z \approx \varepsilon_y \ll 1$$

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- Opposite to DNS, model reduction $\rightarrow$ to decrease the computational cost

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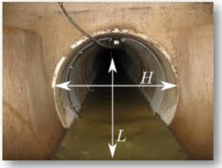
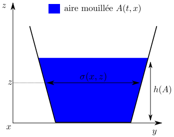
- One can reduce the initial model (Navier-Stokes or Euler equations)
- Opposite to DNS, model reduction $\rightarrow$ to decrease the computational cost
- Some applications :


Outline

SV equations

- 3D-1D model reduction for closed water pipes/channels/rivers



$$\left\{\begin{array}{l} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(x,A) \right) = g I_2(x,A) \end{array}\right.$$

with

$A(t,x)$, $Q(t,x)$, g, $h = \eta - d$	:	wet area, discharge, gravity
$I_1(x,A) = \displaystyle\int_d^\eta \sigma(x,z)(\eta - z)dz$	:	hydrostatic pressure
$I_2(x,A) = \displaystyle\int_d^\eta \frac{\partial}{\partial x}\sigma(x,z)(\eta - z)dz$	:	hydrostatic pressure source

C. Bourdarias, **M. Ersoy**, S. Gerbi.
A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.
International Journal on Finite Volumes, 2009.

C. Bourdarias, **M. Ersoy**, S. Gerbi.
A kinetic scheme for transient mixed flows in non uniform closed pipes : a global manner to upwind all the source terms.
Journal of Scientific Computing, 2011.

C. Bourdarias, **M. Ersoy**, S. Gerbi.
Unsteady mixed flows in non uniform closed water pipes : a Full Kinetic Appraoch.
Numerische Mathematik, 2014.

M. Ersoy.
Dimension reduction for incompressible pipe and open channel flow including friction.
Applications of Mathematics, 2015.

SV equations

- 3D-1D model reduction for closed water pipes/channels/rivers
- 2D-1D reduction for urban/overland flows including precipitation and recharge

$$\begin{cases} \partial_t h + \partial_x q = S := R - I, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{A} + g\dfrac{h^2}{2} \right) = -gh\partial_x Z + S\dfrac{q}{h} - \left(k_+(R) + k_-(I) + k_0\left(\dfrac{q}{h}\right)\right)\dfrac{q}{h} \end{cases}$$

with $h(t,x)$, $q(t,x)$: water height, discharge
$k_\pm$: friction generated from precipitation and infiltration

where I can be driven by the solution of the Richards' equation.

M. Ersoy, O. Lakkis, P. Townsend.
A Saint-Venant shallow water model for overland flows with precipitation and recharge.
Mathematical and Computational Applications, Natural Sciences, 2020.

J.-B. Clément, **M. Ersoy**, F. Golay, and D. Sous.
Wave-driven Ground- water Flows in Sandy Beaches : A Richards Equation-based Model.
Journal of Coastal Research, 2020

J.-B. Clément, **M. Ersoy**, F. Golay, and D. Sous.
Adaptive discontinuous galerkin method for richards equation.
Topical Problems of Fluid Mechanics, 2020

J.-B. Clément, **M. Ersoy**, F. Golay, and D. Sous.
An adaptive strategy for discontinuous Galerkin simulations of Richards' equation : application to multi-materials dam wetting.
Advances in Water Resources, 2021

SV equations

- 3D-1D model reduction for closed water pipes/channels/rivers
- 2D-1D reduction for urban/overland flows including precipitation and recharge
- 3D-2D reduction for tsunamis propagation

$$\begin{cases} \partial_t h + \mathrm{div}(h\overline{u}) = 0, \\ \partial_t (h\overline{u}) + \mathrm{div}\left(h\overline{u} \otimes \overline{u} + g\dfrac{h^2}{2} I \right) = -gh\nabla Z, \end{cases}$$

with $\overline{u}(t,x) \in \mathbb{R}^2$: depth averaged velocity

K. Pons, **M. Ersoy**.
Adaptive mesh refinement method. Part 1 : Automatic thresholding based on a distribution function.
SEMA SIMAI Springer Series, Partial Differential Equations : Ambitious Mathematics for Real-Life Applications, D. Donatelli and C. Simeoni Editors, 2020

K. Pons, **M. Ersoy** , F. Golay and R. Marcer.
Adaptive mesh refinement method. Part 2 : Application to tsunamis propagation.
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- Tsunamis are water waves that start in the deep ocean : H is huge

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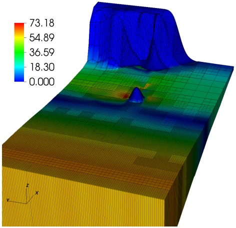
- But, the wavelength λ of the tsunami is huge as well (200 km)
  - Dynamics of tsunamis are "essentially" governed by the shallow water equations.
  - Phase speed of propagation $v_\phi \approx \sqrt{gH}$ (H ocean depth)
  - Use λ instead of L in the derivation $\rightarrow$ shallow water models : justify the use of Saint-Venant equations for some tsunamis.

- Tsunamis are water waves that start in the deep ocean : H is huge
- But, the wavelength λ of the tsunami is huge as well (200 km)
- Tsunami runup onto a complex three dimensional Monai Valley :

	Adap. sim.	Unif. sim.
T_f	30 s	30 s
Nb. blocks	240	240
Nb. cells	8 000-40 000	62 000
Re-mesh. δt	0.25 s	X
CFL	0.5	0.5

Table – Numerical parameters



Numerical water height (coloration is issue from the kinetic energy) at $t = 11.25$ s

[BEG12] K. Pons, M. Ersoy.
Adaptive mesh refinement method. Part 1 : Automatic thresholding based on a distribution function.
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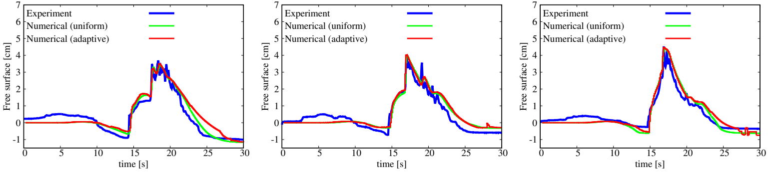

(a) Gauge 1 (b) Gauge 2 (c) Gauge 3

Figure – Free surface results at different positions : experimental data versus numerical simulation with and without mesh adaptivity

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  - Earthquake generated tsunamis, if the magnitude is large enough, hydrostatic models are accurate. Monai Valley flooding is an example (Hokkaido-Nansei-Oki tsunami, 1993, Mw 7,7).

Coming back to the modelling problem : "SVE for certain tsunamis"

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  - Land-slide/subaerial landslide generated tsunamis (depending on landslide thickness, water depth) cannot be represented by hydrostatic models !
    $\rightarrow$ *Glimsdal, Pedersen, Harbitz, Lovholt, Dutykh, Bonneton*, *etc.*
  - dispersions are expected

Parisot and Ersoy's experimental wave generator 
(Malaga, NumHyp 2019)

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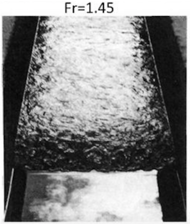

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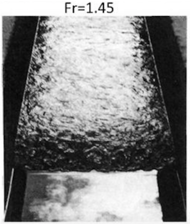

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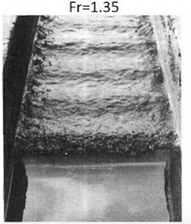

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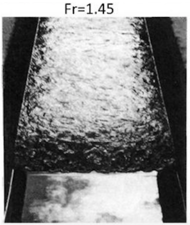

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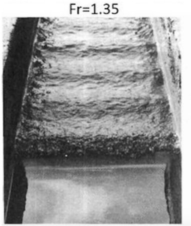

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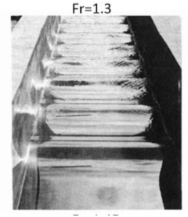

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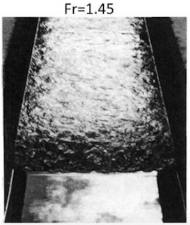

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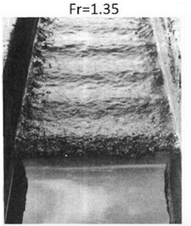

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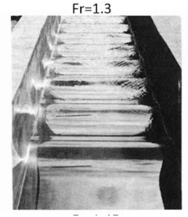

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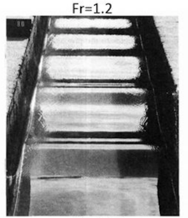

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- Of course, Navier-Stokes equation can deal for both but too costly !

Let $\omega = \dfrac{2\pi}{T}$ be the angular frequency (pulsation) and $k = \dfrac{2\pi}{\lambda}$ wavenumber.

- A wave $\phi(kx - \omega t)$ is characterised by two different characteristic speeds
  - phase velocity $C_p = \dfrac{\omega}{k}$ which corresponds to the displacement of the wave fronts
  - group velocity $C_g = \dfrac{\partial \omega}{\partial k}$ which corresponds to the displacement of the wave's envelope
  - dispersion relation is given by $\omega = C_p k$
- If C_p is constant then the wave is not dispersive.

Dispersive wave

Non dispersive wave

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- According to linear Stokes' theory, noting H the depth, the dispersion relation is

$$\underline{\omega^2 = gk\tanh(kH)}$$

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- at order > 1, $\left(\dfrac{\omega}{k}\right)^2 \approx gH - gk^2H^3 + \dots \rightsquigarrow$ Dispersive models

- Everything starts with Russell's "Wave of translation"

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion ; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation". **John** Scott Russell

- Everything starts with Russell's "Wave of translation"
- Proof of the stability of the solitary wave given by Boussinesq (1872)/Korteweg and Gustav de Vries (1895) through a 1D scalar equation : a perfect equilibrium between non-linearities and the dispersive terms

$$u_t + 6uu_x + u_{xxx} = 0$$

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  - **1976** : Green and Naghdi derived the famous 2D fully nonlinear dispersive equations for uneven bottom (1D below)

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}h + \dfrac{\partial}{\partial x}(hu) = 0 \\ \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}\left(hu^2 + \dfrac{h^2}{2F_r^2}\right) + \mu\dfrac{\partial}{\partial x}\left(\dfrac{h^3}{3}\mathcal{D}(u)\right) = 0 \end{array}\right.$$ with

$$\mathcal{D}(u) = \left(\frac{\partial}{\partial x}u\right)^2 - \frac{\partial}{\partial t}\frac{\partial}{\partial x}u - u\frac{\partial}{\partial x}\frac{\partial}{\partial x}u$$

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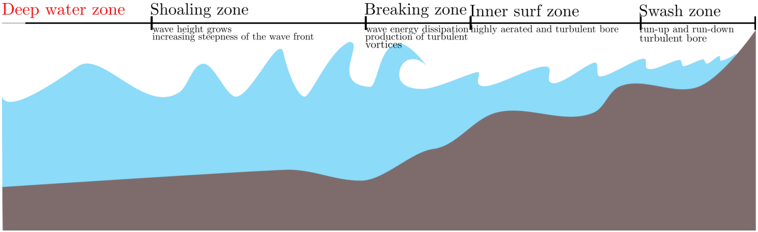
  - 1976 : Green and Naghdi derived the famous 2D fully nonlinear dispersive equations for uneven bottom.
  - Nowadays : Marche, Lannes, Bonneton, Durand, Cienfuegos, Dutykh, Gavrilyuk, Richard, Sainte-Marie, . . . proposed several improvements

- SGN based models are certainly the most appropriate ones for dispersive waves.[a]

a. Lannes, Marche, Durand, Bonneton, Cienfuegos, Dutykh, Gavrilyuk,. . .

- SGN based models are certainly the most appropriate ones for dispersive waves.
- But, dispersive and non dispersive waves can coexist during the Tsunami's life ...
  - Deep water zone : Depth-averaged models hydrostatic and non-hydrostatic models are valid but dispersive codes boosts the CPU times and memory requirements



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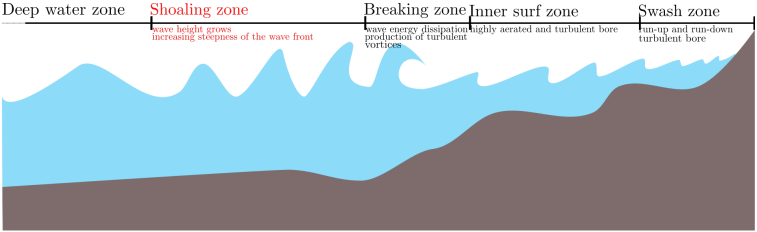
  - Shoaling zone : hydrostatic models are (often) not valid in this zone, leading to an incorrect growth of the wave, yielding to an incorrect prediction of the location of wave breaking



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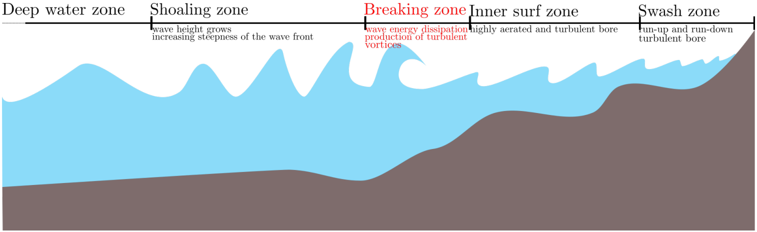
  - Breaking zone : hydrostatic models (SVE) can accurately reproduce broken wave dissipation and swash oscillations without any ad-hoc parametrisation



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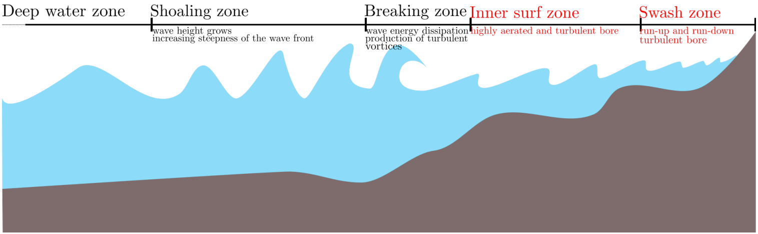
  - Inner surf and swash zones : predominant non-linearities (SVE)



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- Dissipative models are required [a] : "switching from one model to an other"

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a. Lannes, Marche, Durand, Bonneton, Cienfuegos, Dutykh, Gavrilyuk, Pons, ...

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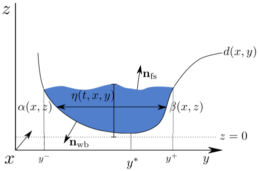
  - 2D models for rivers/channels can be used but costly in the large scale simulation
  - Hydrostatic 1D section-averaged models are well-mastered
  - Non-hydrostatic 1D section-averaged have not yet been derived
    $\rightarrow$ toward the first full non-linear and weakly dispersive section-averaged model

Incompressible Euler equations

$$\begin{array}{rcl} \mathrm{div}(\rho_0 \boldsymbol{u}) & = & 0, \\ \dfrac{\partial}{\partial t}(\rho_0 \boldsymbol{u}) + \mathrm{div}(\rho_0 \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \rho_0 \boldsymbol{F} & = & 0 \end{array}$$



with

$\boldsymbol{u} = (u,v,w)$	:	velocity field
ρ_0	:	density
$\boldsymbol{F} = (0,0,-g)$	:	external force
p	:	pressure

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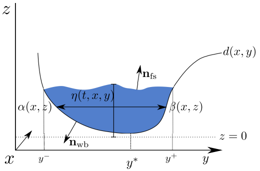


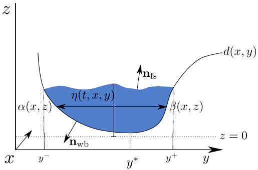
with

$\boldsymbol{u} = (u, v, w)$	:	velocity field
ρ_0	:	density
$\boldsymbol{F} = (0, 0, -g)$	:	external force
p	:	pressure

completed with the irrotational relations

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \ .$$

Incompressible and irrotational Euler equations



$$\begin{array}{rcl} \mathrm{div}(\rho_0 \boldsymbol{u}) & = & 0, \\ \dfrac{\partial}{\partial t}(\rho_0 \boldsymbol{u}) + \mathrm{div}(\rho_0 \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \rho_0 \boldsymbol{F} & = & 0 \end{array}$$

- free surface kinematic boundary condition,

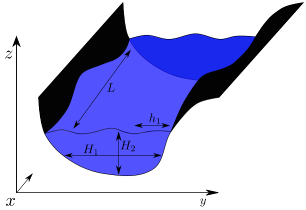
$$\boldsymbol{u} \cdot \boldsymbol{n}_{\mathrm{fs}} = \frac{\partial}{\partial t} \boldsymbol{m} \cdot \boldsymbol{n}_{\mathrm{fs}} \text{ and } p(t, \boldsymbol{m}) = p_0, \ \forall \boldsymbol{m}(t,x,y) = (x,y,\eta(t,x,y)) \in \Gamma_{\mathrm{fs}}(t,x)$$

- no-penetration condition on the wet boundary

$$\boldsymbol{u} \cdot \boldsymbol{n}_{\mathrm{wb}} = 0, \ \forall \boldsymbol{m}(x,y) = (x,y,d(x,y)) \in \Gamma_{\mathrm{wb}}(x)$$

Let us define the dispersive parameters

- $\mu_1 = \dfrac{h_1^2}{L^2}$
- $\mu_2 = \dfrac{H_2^2}{L^2}$,



such that

$$h_1 < H_1 = H_2 \ll L, \text{i.e. } \mu_1 < \mu_2^2$$

where

H_1	:	characteristic scale of channel width
h_1	:	characteristic wave-length in the transversal direction
H_2	:	characteristic water depth
$F_r = \dfrac{U}{\sqrt{gH_2}}$	:	Froude's number
$T = \dfrac{L}{U}$	:	characteristic time
$\mathcal{P} = U^2$	:	characteristic pressure
X	:	characteristic length of x

Then, define the dimensionless variables

$$
\begin{array}{llll}
\widetilde{x} = \dfrac{x}{L}, & \widetilde{P} = \dfrac{P}{\mathcal{P}}, & & \widetilde{\varphi} = \dfrac{\varphi}{h_1}, \\[2ex]
\widetilde{y} = \dfrac{y}{h_1}, & \widetilde{u} = \dfrac{u}{U}, & & \widetilde{d} = \dfrac{d}{H_2}, \\[2ex]
\widetilde{z} = \dfrac{z}{H_2}, & \widetilde{v} = \dfrac{v}{V} = \dfrac{v}{\sqrt{\mu_1}U}, & & \widetilde{\eta} = \dfrac{\eta}{H_2} \ . \\[2ex]
\widetilde{t} = \dfrac{t}{T}, & \widetilde{w} = \dfrac{w}{W} \ = \dfrac{w}{\sqrt{\mu_2}U} \ . & &
\end{array}
$$

We get

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} &= 0 \\
\mu_1 \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) + \frac{\partial P}{\partial y} &= 0 \\
\mu_2 \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) + \frac{\partial P}{\partial z} &= -\frac{1}{{F_r}^2}
\end{aligned}$$

and

$$\frac{\partial u}{\partial y} = \mu_1 \frac{\partial v}{\partial x}, \;\; \mu_1 \frac{\partial v}{\partial z} = \mu_2 \frac{\partial w}{\partial y}, \;\; \frac{\partial u}{\partial z} = \mu_2 \frac{\partial w}{\partial x} \;.$$

Remark I : why $\mu_1 \neq \mu_2$?

$\mu_1 = \mu_2 \Rightarrow$ no analytical expression of the asymptotic terms.

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  $\Rightarrow w(t,x,z) = -\left(\int_d^z u_{|z=d}(t,x)\,dz\right)_x + O(\mu)$
- $\Rightarrow u(t,x,z) = f_1(\bar{u}(t,x)) + \mu f_2(z, \bar{u}(t,x), d(x)) + O(\mu^2)$ where $\bar{u}(t,x) = f_3(u_{|z=d})\ldots$

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Therefore, we assume $\mu_1 < \mu_2$.

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$$\langle u \rangle (t,x,z) = \overline{u}(t,x) + \mu_2 f(\overline{u}(t,x), \Omega(t,x)) + O(\mu_2^2)$$

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$$\int_{\partial\Omega(t,x)} \left(\frac{\partial}{\partial t}\boldsymbol{m} + u\frac{\partial}{\partial x}\boldsymbol{m} - \boldsymbol{v}\right) \cdot \boldsymbol{n}\, ds = 0$$

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  - Introduce wet region indicator function Φ which satisfies

$$\frac{\partial}{\partial t}\Phi + \frac{\partial}{\partial x}(\Phi u) + \mathrm{div}_{y,z}\left[\Phi\boldsymbol{v}\right] = 0 \text{ on } \Omega(t) = \bigcup_{0\leq x\leq 1}\Omega(t,x)$$

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    where $\boldsymbol{v} = (v, w)$.
  - Section-average equations using the approximation

$$\begin{array}{rcl} u(t,x,y,z) & = & \bar{u}(t,x) + \mu_2 B_0(\bar{u},x,z) + O(\mu_2^2) \\ \eta(t,x,y) & = & \bar{\eta}(t,x) + O(\mu_1) \\ P(t,x,y,z) & = & P_{\mathrm{h}}(t,x,z) + \mu_2 P_{\mathrm{nh}}(t,x,z) + O(\mu_2^2) \end{array}$$

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}A + \frac{\partial}{\partial x}Q = 0 \\ \frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + I_1(x,A)\right) + \mu_2\frac{\partial}{\partial x}(\mathcal{D}(u)G(A,x)) = I_2(x,A) \\ +\mu_2\mathcal{G}(u,S,\sigma) + O(\mu_2^2) \end{array}\right.$$

where

$A = \int_{\Omega(t,x)} dy\,dz$	:	wet area
$Q = A(t,x)u(t,x)$	:	discharge
$I_1 = \int_{\Omega(t,x)} \frac{\eta(t,x)-z}{F_r^2}\sigma(x,z)\,dy\,dz$	:	hydro. press.
$I_2 = -\int_{y^-(t,x)}^{y^+(t,x)} \frac{h(t,x)}{{F_r}^2}\frac{\partial}{\partial x}d(x,y)\,dy$	:	hydro. press. source

- Debyaoui, Ersoy. Asymptotic Analysis, 2020

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}Q = 0 \\ \dfrac{\partial}{\partial t}Q + \dfrac{\partial}{\partial x}\left(\dfrac{Q^2}{A} + I_1(x,A)\right) + \mu_2 \dfrac{\partial}{\partial x}(\mathcal{D}(u)G(A,x)) = I_2(x,A) \\ +\mu_2 \mathcal{G}(u,S,\sigma) + O(\mu_2^2) \end{array}\right.$$

where

$$\mathcal{D}(u) = \left(\frac{\partial}{\partial x}u\right)^2 - \frac{\partial}{\partial t}\frac{\partial}{\partial x}u - u\frac{\partial}{\partial x}\frac{\partial}{\partial x}u$$

and

$$G(A,x) = \int_{d^*(x)}^{\eta} \sigma(x,z) \int_z^{\eta} \frac{S(x,s)}{\sigma(x,s)}\, ds\, dz$$

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where

$$\begin{aligned} \mathcal{G}(u,S,\sigma) &= \int_z^\eta \frac{u^2}{\sigma(x,s)}\left(\frac{\dfrac{\partial}{\partial x}S(x,s)\dfrac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)} - \frac{\partial}{\partial x}\frac{\partial}{\partial x}S(x,s)\right) \\ &\quad + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\frac{S(x,s)\dfrac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)^2} \\ &\quad - \left(\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u\right)\frac{\dfrac{\partial}{\partial x}S(x,s)}{\sigma(x,s)}\,ds \end{aligned}.$$

$$
\begin{cases}
\dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}Q = 0 \\
\dfrac{\partial}{\partial t}Q + \dfrac{\partial}{\partial x}\left(\dfrac{Q^2}{A} + I_1(x, A)\right) + \mu_2 \dfrac{\partial}{\partial x}(\mathcal{D}(u)G(A, x)) = I_2(x, A) \\
+\mu_2 \mathcal{G}(u, S, \sigma) + O(\mu_2^2)
\end{cases}
$$

Setting $\sigma = 1$, $d = 1$,

- $A = h$
- $S(x, z) \equiv S(z) \Rightarrow \mathcal{G} = 0$ and $I_2 = 0$
- $G = \dfrac{h^3}{3}$
- $I_1 = \dfrac{h^2}{2F_r^2}$

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}Q = 0 \\ \dfrac{\partial}{\partial t}Q + \dfrac{\partial}{\partial x}\left(\dfrac{Q^2}{A} + I_1(x,A)\right) + \mu_2 \dfrac{\partial}{\partial x}(\mathcal{D}(u)G(A,x)) = I_2(x,A) \\ +\mu_2 \mathcal{G}(u,S,\sigma) + O(\mu_2^2) \end{array}\right.$$

we recover the classical SGN equations on flat bottom

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}h + \dfrac{\partial}{\partial x}(hu) = 0 \\ \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}\left(hu^2 + \dfrac{h^2}{2F_r^2}\right) + \mu_2 \dfrac{\partial}{\partial x}\left(\dfrac{h^3}{3}\mathcal{D}(u)\right) = O(\mu_2^2) \end{array}\right.$$

where

$$\mathcal{D}(u) = \left(\frac{\partial}{\partial x}u\right)^2 - \frac{\partial}{\partial t}\frac{\partial}{\partial x}u - u\frac{\partial}{\partial x}\frac{\partial}{\partial x}u$$

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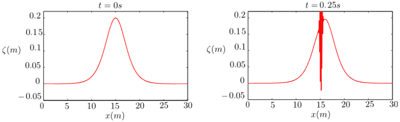
REMARK

Dispersive equation are usually characterised by third order term

$\Downarrow$

time step restriction and may create high frequencies instabilities



Bourdarias, Gerbi, and Ralph Lteif. Computers & Fluids, 156 :283–304, 2017.

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators

$$\mathcal{T}[A,d,\sigma,z](u) = \frac{\partial}{\partial x}(u)\int_z^\eta \frac{S(x,s)}{\sigma(x,s)}\,ds + u\int_z^\eta \frac{1}{\sigma(x,s)}\frac{\partial}{\partial x}S(x,s)\,ds\ ,$$

and

$$\begin{aligned}\mathcal{G}[A,d,\sigma,z](u) \quad = \quad & \int_z^\eta 2\left(\frac{\partial}{\partial x}u\right)^2\frac{S(x,s)}{\sigma(x,s)}+ \\ & \frac{u^2}{\sigma(x,s)}\left(\frac{\frac{\partial}{\partial x}S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)} - \frac{\partial}{\partial x}\frac{\partial}{\partial x}S(x,s)\right) \\ & +\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\frac{S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)^2}\,ds\end{aligned}$$

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
- Define the averaged linear $\overline{\mathcal{T}}$ and the quadratic $\overline{\mathcal{Q}}$ operators

$$\overline{\mathcal{T}}[A,d,\sigma](u,\psi) \quad = \quad \int_{d^*(x)}^{\eta} \psi\mathcal{T}[A,d,\sigma,z](u)\ dz$$

and

$$\overline{\mathcal{G}}[A,d,\sigma](u,\psi) \quad = \quad \int_{d^*(x)}^{\eta} \psi\mathcal{G}[A,d,\sigma,z](u)\ dz$$

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
- Define the averaged linear $\overline{\mathcal{T}}$ and the quadratic $\overline{\mathcal{Q}}$ operators
- Define the operators $\mathcal{L}$ and $\mathcal{Q}$

$$\mathbb{L}[A,d,\sigma](u) \quad = \quad A\mathcal{L}[A,d,\sigma]\left(\frac{u}{A}\right)$$

and

$$\mathcal{Q}[A,d,\sigma](u) \quad = \quad \frac{1}{A}\left[\frac{\partial}{\partial x}\left(\overline{\mathcal{G}}[A,d,\sigma]\,(u,\sigma)\right) - \overline{\mathcal{G}}[A,d,\sigma]\left(u, \frac{\partial}{\partial x}\sigma\right)\right]$$

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
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- and finally the operator $\mathbb{L}$

$$\mathbb{L}[A, d, \sigma](u) \quad = \quad A\mathcal{L}[A, d, \sigma]\left(\frac{u}{A}\right)$$

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- and finally the operator $\mathbb{L}$
- Reformulated model

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}(Au) = 0 \\ \left(I_d - \mu_2 \mathbb{L}[A,d,\sigma]\right)\left(\dfrac{\partial}{\partial t}(Au) + \dfrac{\partial}{\partial x}\left(Au^2\right)\right) + \dfrac{\partial}{\partial x}I_1(x,A) \\ +\mu_2 A\mathcal{Q}[A,d,\sigma](u) = I_2(x,A) + O(\mu_2^2) \end{array}\right.$$

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Remark

Inverting $I_d-\mu_2\mathbb{L}[A,d,\sigma]$ $\Rightarrow$ no third order term $\Rightarrow$ more stable formulation

- Bonneton, Barthélemy, Chazel, Cienfuegos, Lannes, Marche, and Tissier. European Journal of Mechanics-B/Fluids, 2011
- Debyaoui, Ersoy. Recent Advances in Numerical Methods for Hyperbolic PDE Systems. SEMA SIMAI Springer Series, 2021

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
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- and finally the operator $\mathbb{L}$
- Reformulated model

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}(Au) = 0 \\ (I_d - \mu_2\kappa\mathbb{L}[A,d,\sigma])\left(\dfrac{\partial}{\partial t}(Au) + \dfrac{\partial}{\partial x}(Au^2) + \dfrac{\kappa-1}{\kappa}\left(\dfrac{\partial}{\partial x}I_1 - I_2\right)\right) \\ +\dfrac{1}{\kappa}\left(\dfrac{\partial}{\partial x}I_1 - I_2\right) + \mu_2 A\mathcal{Q}[A,d,\sigma](u) = O(\mu_2^2) \end{array}\right.$$

Remark

A consistent one-parameter $\kappa > 0$ family (up to order $O(\mu_2^2)$) can be introduced to improve the frequency dispersion.

- Bonneton, Barthélemy, Chazel, Cienfuegos, Lannes, Marche, and Tissier. European Journal of Mechanics-B/Fluids, 2011
- Debyaoui, Ersoy. Recent Advances in Numerical Methods for Hyperbolic PDE Systems. SEMA SIMAI Springer Series, 2021

Theorem

Let α,β and $d \in C_b^\infty$ and $A \in W^{1,\infty}(\mathbb{R})$ such that $\inf_{x\in\mathbb{R}} A \geq A_0 > 0$. Then the operator

$$\mathbb{T} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is well-defined, one-to-one and onto.

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is well-defined, one-to-one and onto.

- Let $\mu_2 \in (0, 1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$ the space $H^1(\mathbb{R})$ endowed with the norm

$$\| u \|_{\mu_2}^2 = \| u \|_2^2 + \mu_2 \| u_x \|_2^2$$

THEOREM

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is well-defined, one-to-one and onto.

- Let $\mu_2 \in (0,1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$
- Define the bilinear form $a(u,v)$

$$a(u,v) = (A\mathbb{T}u, v) = (Au, v) +$$

$$\mu_2 \left(A \left(\frac{A}{\sqrt{3}u_x} - \frac{\sqrt{3}}{2} d_x u \right), \left(\frac{A}{\sqrt{3}v_x} - \frac{\sqrt{3}}{2} d_x v \right) \right) + (A d_x u, d_x v)$$

Theorem

Let α,β and $d \in C_b^\infty$ and $A \in W^{1,\infty}(\mathbb{R})$ such that $\inf_{x\in\mathbb{R}} A \geq A_0 > 0$. Then the operator

$$\mathbb{T} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is well-defined, one-to-one and onto.

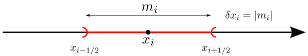
- Let $\mu_2 \in (0, 1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$
- Define the bilinear form $a(u, v)$
- Lax-Milgram theorem

$$\exists!\ u \in H^1_{\mu_2}(\mathbb{R})\ ;\ a(u, v) = (f, v),\ \forall v \in H^1_{\mu_2}(\mathbb{R}),\ f \in L^2(\mathbb{R})$$

$$\Downarrow$$

$$\exists!\ u \in H^1_{\mu_2}(\mathbb{R})\ ;\ \mathbb{T}u = f$$

- From definition of $\mathbb{T}$, we get $u_{xx} = g(A, u, d, \sigma) \in L^2(\mathbb{R}) \Rightarrow u \in H^2(\mathbb{R})$.



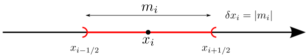
We consider a classical Finite Volume scheme, $\boldsymbol{U} = (A, Q)$

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\delta t^n}{\delta x} \left(\boldsymbol{F}_{i+1/2}(\boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) - \boldsymbol{F}_{i-1/2}(\boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n) \right)$$

where $\boldsymbol{F}_{i\pm 1/2} \approx \dfrac{1}{\delta t^n} \displaystyle\int_{m_i} \boldsymbol{F}(\boldsymbol{U}(t, x_{i+1/2}))\, dx$ is a Finite volume solver,

with

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} Au \\ Au^2 + \dfrac{\kappa - 1}{\kappa} \left(I_1 -'' \displaystyle\int I_2'' \right) \end{pmatrix}$$



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where $\boldsymbol{F}_{i\pm 1/2} \approx \dfrac{1}{\delta t^n}\displaystyle\int_{m_i} \boldsymbol{F}(\boldsymbol{U}(t, x_{i+1/2}))\, dx$ is a Finite volume solver, for instance, with upwind technique to deal with source term

$$\boldsymbol{F}_{i\pm 1/2} = \frac{\boldsymbol{F}(\boldsymbol{U}) + \boldsymbol{F}(\boldsymbol{V})}{2} - \frac{s_i^n}{2}(\boldsymbol{V} - \boldsymbol{U})$$

with

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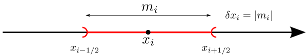
- Bourdarias, Ersoy, Gerbi. Journal of Scientific Computing, 2011



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$$-\frac{\delta t^n}{\delta x}\left(\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1} \boldsymbol{D}^n\right)_i$$

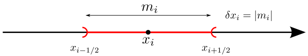
with

$$(\boldsymbol{D}^n)_i = \boldsymbol{D}_{i+1/2}(\boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) - \boldsymbol{D}_{i-1/2}(\boldsymbol{U}_{i-2}^n, \boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n)$$

where $\boldsymbol{D}_{i\pm1/2}$ and $\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1}$ are the centred approximation of

$$\mathcal{D} = \frac{1}{\kappa}\left(\frac{\partial}{\partial x} I_1 - I_2\right) + \mu_2 A \mathcal{Q} \text{ and } \left[(I_d - \mu_2 \mathbb{L})\right]^{-1}$$

Numerical scheme :



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$$-\frac{\delta t^n}{\delta x}\left(\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1} \boldsymbol{D}^n\right)_i$$

Theorem

The numerical scheme is stable under the classical CFL condition,

$$\max_{\lambda \in \mathrm{Sp}(D_{\boldsymbol{U}} \boldsymbol{F}(\boldsymbol{U}))} |\lambda| \frac{\delta t^n}{\delta x} \leqslant 1 \ .$$

- Debyaoui, Ersoy. Recent Advances in Numerical Methods for Hyperbolic PDE Systems. SEMA SIMAI Springer Series, 2021

- Influence of the Section Variation ($N = 5000$ cells) :
  $\sigma(x;\varepsilon) = \beta(x;\varepsilon) - \alpha(x;\varepsilon)$ with
  $\beta = \dfrac{1}{2} - \dfrac{\varepsilon}{2}\exp\left(-\varepsilon^2\left(x - L/2\right)^2\right)\right)$ and $\alpha = -\beta$

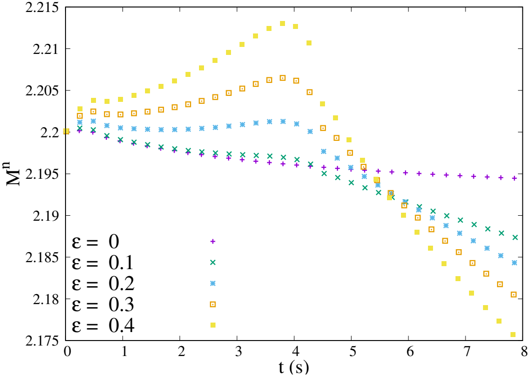

Figure – $M^n := \max\limits_{x \in [0, L_c]} (h_i^n)$

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- Numerical order for $\varepsilon = 0$

	$\lVert \eta_{\text{num}} - \eta_{exact} \rVert_2$	$\lVert \eta_{\text{num}} - \eta_{exact} \rVert_\infty$
Order	0.53	0.58

- Numerical order for $\varepsilon = 0.4$ (reference solution obtained with $N = 10000$ cells)

	$\lVert \eta_{\text{num}} - \eta_{\text{ref}} \rVert_2$	$\lVert \eta_{\text{num}} - \eta_{\text{ref}} \rVert_\infty$
Order	0.64	0.56

- Comparison with the NLSW and the exact solution

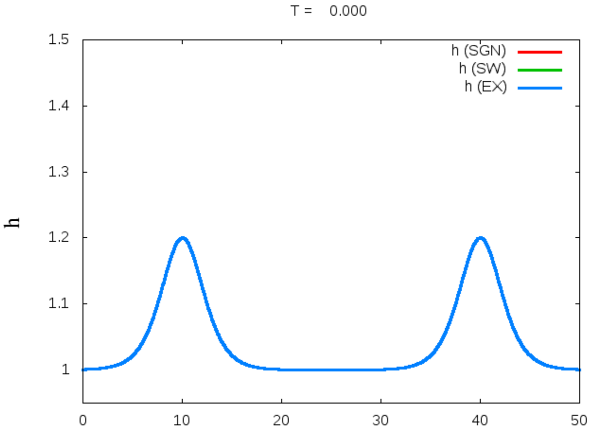
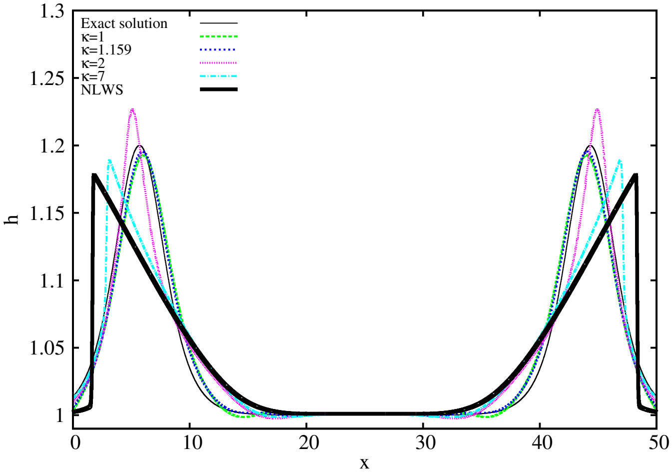


Figure – $\sigma = 1$, $d = 1$, $N = 1000$, $CFL = 0.95$, $T_f = 10$ and $\kappa = 1.159$

- Comparison with the NLSW and the exact solution
- Influence of κ : toward a dissipative shallow water model



(a) Solutions at time $T_f = 10$

Outline

Outline

Thank you

for your

attention