Unsteady mixed flows in closed water pipes.

A well-balanced finite volume scheme

Christian Bourdarias, **Mehmet Ersoy** and Stéphane Gerbi

LAMA, Université de Savoie, Chambéry, France

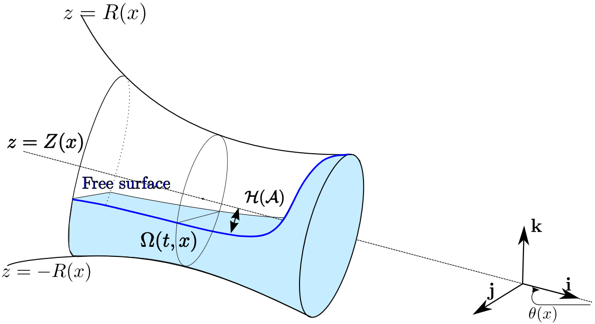
2nd Workshop Mathematics and Oceanography
Montpellier, 1-2-3 february 2010.

1. Modelisation: the pressurized and free surface flows model
   - Previous works about mixed model
   - The free surface model
   - The pressurized model
   - The PFS-model : a natural coupling
2. Finite Volume discretisation
   - Discretisation of the space domain
   - Explicit first order VFRoe scheme
     1. The Case of a non transition point
     2. The Case of a transition point
     3. Update of the cell state
     4. Approximation of the convection matrix
3. Numerical experiments
4. Conclusion and perspectives

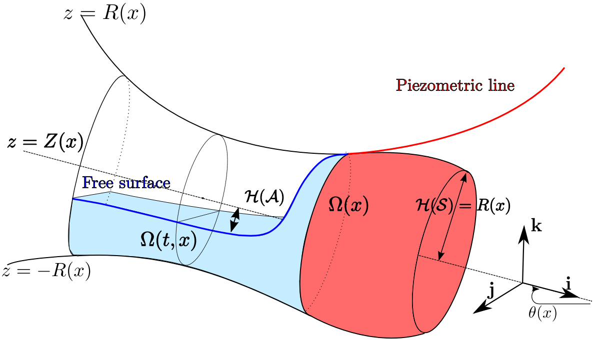
Outline

1. Modelisation: the pressurized and free surface flows model
   - Previous works about mixed model
   - The free surface model
   - The pressurized model
   - The PFS-model : a natural coupling
2. Finite Volume discretisation
   - Discretisation of the space domain
   - Explicit first order VFRoe scheme
     1. The Case of a non transition point
     2. The Case of a transition point
     3. Update of the cell state
     4. Approximation of the convection matrix
3. Numerical experiments
4. Conclusion and perspectives

- Free surface (FS) area : only a part of the section is filled. Incompressible?. . .

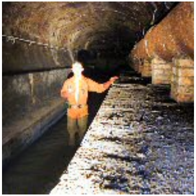


- Free surface (FS) area : only a part of the section is filled. Incompressible?...
- Pressurized (P) area : the section is completely filled. Compressible? Incompressible?...

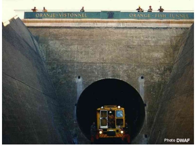


Some closed pipes

a forced pipe



a sewer in Paris



The Orange-Fish Tunnel

Storm Water Overflow, Minnesota

http://www.sewerhistory.org/grfx/misc/disaster.htm

Free surface flows

Saint-Venant equations for open channels

Pressurized flows : Allievi equation

$$\begin{aligned}
\frac{\partial P}{\partial t} + \frac{c^2}{g\,A}\frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + g\,A\frac{\partial P}{\partial x} &= -\alpha\, Q\,|Q|
\end{aligned}$$

A lot of terms have been neglected: no conservative form

Goal :

Free surface flows

Saint-Venant equations for open channels

Pressurized flows : Allievi equation

$$\begin{aligned}
\frac{\partial P}{\partial t} + \frac{c^2}{g\,A}\frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + g\,A\frac{\partial P}{\partial x} &= -\alpha\, Q\,|Q|
\end{aligned}$$

A lot of terms have been neglected: no conservative form

Goal :

1-to write a model for pressurized flows “close to” Saint-Venant equations

Free surface flows

Saint-Venant equations for open channels

Pressurized flows : Allievi equation

$$
\begin{aligned}
\frac{\partial P}{\partial t} + \frac{c^2}{g\,A}\frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + g\,A\frac{\partial P}{\partial x} &= -\alpha\, Q\,|Q|
\end{aligned}
$$

A lot of terms have been neglected: no conservative form

Goal :

2-to get a single model for pressurized and free surface flows

Free surface flows

Saint-Venant equations for open channels

Pressurized flows : Allievi equation

$$\begin{aligned}\frac{\partial P}{\partial t} + \frac{c^2}{g\,A}\frac{\partial Q}{\partial x} &= 0\\ \frac{\partial Q}{\partial t} + g\,A\frac{\partial P}{\partial x} &= -\alpha\,Q\,|Q|\end{aligned}$$

A lot of terms have been neglected: no conservative form

Goal :

3-to take into account depression phenomena

Free surface flows

Saint-Venant equations for open channels

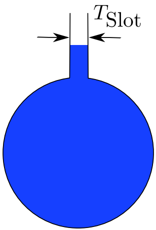
Pressurized flows : Allievi equation

$$
\begin{aligned}
\frac{\partial P}{\partial t} + \frac{c^2}{g\,A}\frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + g\,A\frac{\partial P}{\partial x} &= -\alpha\, Q\,|Q|
\end{aligned}
$$

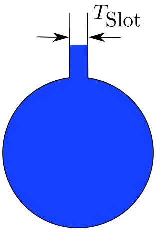
A lot of terms have been neglected: no conservative form

Goal :

as follows : click



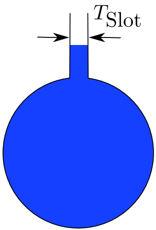
- Preissmann (1961), Cunge and Wenger (1965), Song and Cardle (1983)
- Garcia-Navarro *et al.* (1994) , Zech *et al.* (1997): finite difference and characteristics method or Roe's method
- Baines *et al.* (1992), Tseng (1999): Roe scheme on finite volume



- Preissmann (1961), Cunge and Wenger (1965), Song and Cardle (1983)
- Garcia-Navarro *et al.* (1994) , Zech *et al.* (1997): finite difference and characteristics method or Roe's method
- Baines *et al.* (1992), Tseng (1999): Roe scheme on finite volume

Good behavior

We used only Saint-Venant equations, very easy to solve ...



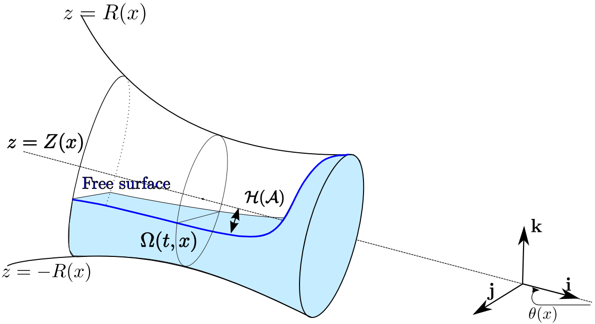
- Preissmann (1961), Cunge and Wenger (1965), Song and Cardle (1983)
- Garcia-Navarro *et al.* (1994) , Zech *et al.* (1997): finite difference and characteristics method or Roe's method
- Baines *et al.* (1992), Tseng (1999): Roe scheme on finite volume

Bad behavior

- sound speed $\simeq \sqrt{S/T_{slot}}$
- water-hammer are not well computed
- depression in pressurized flows : free surface transition

- Hamam et McCorquodale (82): “rigid water column approach”; a water column follows a dilatation-compression process .
- Trieu Dong (1991) Finite difference method : on each cell conservativity of mass and momentum are written depending on the state.
- Musandji Fuamba (2002) : Saint-Venant (free surface) and compressible fluid (pressurized flow); finite difference and characteristics method.
- Vasconcelos, Wright and Roe (2006). Two Pressure Approach and Roe scheme; the overpressure or depression computed via the dilatation of the pipe.

$$\begin{aligned} \mathrm{div}(\mathbf{U}) &= 0 \\ \partial_t(\mathbf{U}) + \mathbf{U}\cdot\nabla\mathbf{U} + \nabla p &= F \end{aligned}$$



The domain $\Omega_F(t)$ of the flow at time t : the union of sections $\Omega(t,x)$ orthogonal to some plane curve $\mathcal{C}$ lying in $(O,\mathbf{i},\mathbf{k})$ following main flow axis.
$\omega = (x, 0, b(x))$ in the cartesian reference frame $(O,\mathbf{i},\mathbf{j},\mathbf{k})$ where $\mathbf{k}$ follows the vertical direction; $b(x)$ is then the elevation of the point $\omega(x,0,b(x))$ over the plane $(O,\mathbf{i},\mathbf{j})$
Curvilinear variable defined by:

$$X = \int_{x_0}^{x} \sqrt{1 + (b'(\xi))^2} d\xi$$

where x_0 is an arbitrary abscissa. $Y = y$ and we denote by Z the $\mathbf{B}$-coordinate of any fluid particle M in the Serret-Frenet reference frame $(\mathbf{T},\mathbf{N},\mathbf{B})$ at point $\omega(x,0,b(x))$.

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and take $\epsilon = 0$ in the Euler curvilinear equations,
3. approximation :$\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. the conservative variables $A(t,X)$: the wet area, $Q(t,X)$ the discharge defined by

$$A(t,X) = \int_{\Omega(t,X)} dYdZ, \quad Q(t,X) = A(t,X)\overline{U}$$

$$\overline{U}(t,X) = \frac{1}{A(t,X)} \int_{\Omega(t,X)} U(t,X)\; dYdZ.$$

[GP01] *J.-F. Gerbeau, B. Perthame* Derivation of viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation. *Discrete and Continuous Dynamical Systems*, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

[F07] *F. Marche* Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European Journal of Mechanic B / Fluid*, 26 (2007), 49–63.

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and take $\epsilon = 0$ in the Euler curvilinear equations,
3. approximation :$\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. the conservative variables $A(t,X)$: the wet area, $Q(t,X)$ the discharge defined by

$$A(t,X) = \int_{\Omega(t,X)} dYdZ, \quad Q(t,X) = A(t,X)\overline{U}$$

$$\overline{U}(t,X) = \frac{1}{A(t,X)} \int_{\Omega(t,X)} U(t,X)\, dYdZ.$$

[GP01] *J.-F. Gerbeau, B. Perthame* Derivation of viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation. *Discrete and Continuous Dynamical Systems*, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

[F07] *F. Marche* Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European Journal of Mechanic B / Fluid*, 26 (2007), 49–63.

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and take $\epsilon = 0$ in the Euler curvilinear equations,
3. approximation :$\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. the conservative variables $A(t,X)$: the wet area, $Q(t,X)$ the discharge defined by

$$A(t,X) = \int_{\Omega(t,X)} dYdZ, \quad Q(t,X) = A(t,X)\overline{U}$$

$$\overline{U}(t,X) = \frac{1}{A(t,X)} \int_{\Omega(t,X)} U(t,X)\; dYdZ.$$

[GP01] *J.-F. Gerbeau, B. Perthame* Derivation of viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation. *Discrete and Continuous Dynamical Systems*, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

[F07] *F. Marche* Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European Journal of Mechanic B / Fluid*, 26 (2007), 49–63.

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and take $\epsilon = 0$ in the Euler curvilinear equations,
3. approximation :$\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. the conservative variables $A(t,X)$: the wet area, $Q(t,X)$ the discharge defined by

$$A(t,X) = \int_{\Omega(t,X)} dYdZ, \quad Q(t,X) = A(t,X)\overline{U}$$

$$\overline{U}(t,X) = \frac{1}{A(t,X)} \int_{\Omega(t,X)} U(t,X)\; dYdZ.$$

[GP01] *J.-F. Gerbeau, B. Perthame* Derivation of viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation. *Discrete and Continuous Dynamical Systems*, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

[F07] *F. Marche* Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European Journal of Mechanic B / Fluid*, 26 (2007), 49–63.

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and take $\epsilon = 0$ in the Euler curvilinear equations,
3. approximation :$\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. the conservative variables $A(t,X)$: the wet area, $Q(t,X)$ the discharge defined by

$$A(t,X) = \int_{\Omega(t,X)} dYdZ, \quad Q(t,X) = A(t,X)\overline{U}$$

$$\overline{U}(t,X) = \frac{1}{A(t,X)} \int_{\Omega(t,X)} U(t,X)\, dYdZ.$$

[GP01] *J.-F. Gerbeau, B. Perthame* Derivation of viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation. *Discrete and Continuous Dynamical Systems*, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

[F07] *F. Marche* Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European Journal of Mechanic B / Fluid*, 26 (2007), 49–63.

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and take $\epsilon = 0$ in the Euler curvilinear equations,
3. approximation :$\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. the conservative variables $A(t,X)$: the wet area, $Q(t,X)$ the discharge defined by

$$A(t,X) = \int_{\Omega(t,X)} dYdZ, \quad Q(t,X) = A(t,X)\overline{U}$$

$$\overline{U}(t,X) = \frac{1}{A(t,X)} \int_{\Omega(t,X)} U(t,X)\, dYdZ.$$

[GP01] *J.-F. Gerbeau, B. Perthame* Derivation of viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation. *Discrete and Continuous Dynamical Systems*, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

[F07] *F. Marche* Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European Journal of Mechanic B / Fluid*, 26 (2007), 49–63.

$$\left\{\begin{array}{lcl}\partial_t A + \partial_X Q & = & 0 \\ \partial_t Q + \partial_X\left(\dfrac{Q^2}{A} + gI_1(X,A)\cos\theta\right) & = & gI_2(X,A)\cos\theta - gA\sin\theta \\ & & -gA\overline{Z}(X,A)(\cos\theta)'\end{array}\right. \quad (1)$$

$I_1(X,A) = \displaystyle\int_{-R}^{h}(h-Z)\sigma\, dZ$: the hydrostatic pressure term

$I_2(X,A) = \displaystyle\int_{-R}^{h}(h-Z)\partial_X\sigma\, dZ$: the pressure source term

$\overline{Z} = \displaystyle\int_{\Omega(t,X)} Z\, dY\, dZ$: the center of mass

We add the Manning-Strickler friction term of the form

$$S_f(A,U) = K(A)U|U|\,.$$

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{U}) = 0, \tag{2}$$

$$\partial_t(\rho \mathbf{U}) + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{F}, \tag{3}$$

Linearized pressure law:

$$p = p_a + \frac{\rho - \rho_0}{\beta \rho_0}$$

$$c = \frac{1}{\sqrt{\beta \rho_0}} \simeq 1400 m/s$$

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and takes $\epsilon = 0$ in the Euler curvilinear equations,
3. Approximation :$\overline{\rho U} \approx \overline{\rho}\,\overline{U}$ and $\overline{\rho U^2} \approx \overline{\rho}\,\overline{U}\,\overline{U}$.
4. the conservative variables $A(t,X)$: the wet equivalent area, $Q(t,X)$ the equivalent discharge defined by

$$A = \frac{\overline{\rho}}{\rho_0} S\,,\ Q = A\overline{U}$$

$$\overline{U}(t,X) = \frac{1}{S(X)} \int_{S(X)} U(t,X)\ dYdZ.$$

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and takes $\epsilon = 0$ in the Euler curvilinear equations,
3. Approximation :$\overline{\rho U} \approx \overline{\rho}\,\overline{U}$ and $\overline{\rho U^2} \approx \overline{\rho}\,\overline{U}\,\overline{U}$.
4. the conservative variables $A(t,X)$: the wet equivalent area, $Q(t,X)$ the equivalent discharge defined by

$$A = \frac{\overline{\rho}}{\rho_0} S\,,\ Q = A\overline{U}$$

$$\overline{U}(t,X) = \frac{1}{S(X)} \int_{S(X)} U(t,X)\ dYdZ.$$

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and takes $\epsilon = 0$ in the Euler curvilinear equations,
3. Approximation :$\overline{\rho U} \approx \overline{\rho}\,\overline{U}$ and $\overline{\rho U^2} \approx \overline{\rho}\,\overline{U}\,\overline{U}$.
4. the conservative variables $A(t, X)$: the wet equivalent area, $Q(t, X)$ the equivalent discharge defined by

$$A = \frac{\overline{\rho}}{\rho_0} S\,,\; Q = A\overline{U}$$

$$\overline{U}(t, X) = \frac{1}{S(X)} \int_{S(X)} U(t, X)\, dYdZ.$$

1. write the Euler equations in a curvilinear reference frame,
2. $\epsilon = H/L$ with H (the height) and L (the length) and takes $\epsilon = 0$ in the Euler curvilinear equations,
3. Approximation :$\overline{\rho U} \approx \overline{\rho}\,\overline{U}$ and $\overline{\rho U^2} \approx \overline{\rho}\,\overline{U}\,\overline{U}$.
4. the conservative variables $A(t, X)$: the wet equivalent area, $Q(t, X)$ the equivalent discharge defined by

$$A = \frac{\overline{\rho}}{\rho_0} S\,,\; Q = A\overline{U}$$

$$\overline{U}(t, X) = \frac{1}{S(X)} \int_{S(X)} U(t, X)\, dYdZ.$$

$$\begin{cases} \partial_t(A) + \partial_X(Q) &= 0 \\ \partial_t(Q) + \partial_X\left(\dfrac{Q^2}{A} + c^2 A\right) &= -gA\sin\theta - gA\overline{Z}(X,S)(\cos\theta)' \\ & \quad + c^2 A \dfrac{S'}{S} \end{cases} \quad (4)$$

$c^2 A$: the pressure term

$c^2 A \dfrac{S'}{S}$: the pressure source term due to geometry changes

$gA\overline{Z}(X,S)(\cos\theta)'$: the pressure source term due to the curvature

$\overline{Z}$: the center of mass

We add the Manning-Strickler friction term of the form

$$S_f(A,U) = K(A)U|U|$$

State variable

Let E the *state variable* and $\mathbf{S} = \mathbf{S}(A, E)$ the *physical wet area* such that:

$$\mathbf{S} = \begin{cases} S & \text{if} \quad E = 1(P) \\ A & \text{if} \quad E = 0(FS) \end{cases}$$

$$\left\{\begin{array}{ll}\partial_t(A)+\partial_x(Q) & =0\\ \partial_t(Q)+\partial_x\left(\dfrac{Q^2}{A}+p(x,A,E)\right) & =-g\,A\,\dfrac{d}{dx}Z(x)\\ & +Pr(x,A,E)\\ & -G(x,A,E)\\ & -g\,A\,K(x,\mathbf{S})\,u\,|u|\end{array}\right. .$$

- $A=\dfrac{\rho}{\rho_0}S$: wet equivalent area,
- $Q=A\,u$: discharge,
- S the physical wet area.

The pressure is $p(x,A,E)=c^2\,(A-\mathbf{S})+g\,I_1(x,\mathbf{S})\cos\theta$.

- The pressure source term:

$$Pr(x, A, E) = \left(c^2 \left(A/\mathbf{S} - 1\right)\right) \frac{d}{dx} S + g\, I_2(x, \mathbf{S}) \cos\theta,$$

- the $z-$coordinate of the center of mass term:

$$G(x, A, E) = g\, A\, \overline{Z}(x, \mathbf{S}) \frac{d}{dx} \cos\theta,$$

- the friction term:

$$K(x, \mathbf{S}) = \frac{1}{K_s^2\, R_h(\mathbf{S})^{4/3}}.$$

  - $K_s > 0$ is the Strickler coefficient,
  - $R_h(\mathbf{S})$ is the hydraulic radius.

[BEG09] *C. Bourdarias, M. Ersoy and S. Gerbi.* A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme. *IJFV , 2009.*

Mathematical properties

- The PFS system is strictly hyperbolic for $A(t,x) > 0$.
- For smooth solutions, the mean velocity $u = Q/A$ satisfies

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + c^2 \ln(A/S) + g\,\mathcal{H}(S)\cos\theta + g\,Z\right) \\ = -g\,K(x,\mathbf{S})\,u\,|u| \quad .$$

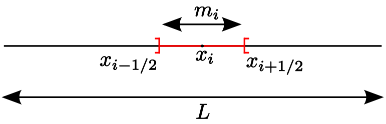
and $u = 0$ reads: $c^2 \ln(A/\mathbf{S}) + g\,\mathcal{H}(\mathbf{S})\cos\theta + g\,Z = 0$.

- It admits a mathematical entropy

$$E(A,Q,S) = \frac{Q^2}{2A} + c^2 A \ln(A/\mathbf{S}) + c^2 S + g\overline{Z}(x,\mathbf{S})\cos\theta + gAZ$$

which satisfies the entropy inequality

$$\partial_t E + \partial_x \left(E\,u + p(x,A,E)\,u\right) = -g\,A\,K(x,\mathbf{S})\,u^2\,|u| \leqslant 0$$



Geometric terms and unknowns are piecewise constant approximations on the cell m_i at time t_n:

- Geometric terms
  - Z_i, S_i, $\cos\theta_i$
- unknowns
  - (A_i^n, Q_i^n), $u_i^n = \dfrac{Q_i^n}{A_i^n}$
- Notation: "unknown" vector
  - $\mathbf{W}_i^n = (Z_i, \cos\theta_i, S_i, A_i^n, Q_i^n)^t$

Outline

Adding the equations $\partial_t Z = 0$, $\partial_t \cos\theta = 0$ and $\partial_t S = 0$, the PFS-model under a non conservative form reads:

$$\partial_t \mathbf{W} + D(\mathbf{W})\partial_X \mathbf{W} = TS(\mathbf{W}) \tag{5}$$

where $\mathbf{W} = (Z, \cos\theta, S, A, Q)^t$

$$TS(\mathbf{W}) = \left(0, 0, 0, 0, -g\,K(X, \mathbf{S})\,\frac{Q|Q|}{A}\right)$$

$$D(\mathbf{W}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ gA & gA\mathcal{H}(\mathbf{S}) & \Psi(\mathbf{W}) & c^2(\mathbf{W}) - u^2 & 2u \end{pmatrix}$$

where $\Psi(\mathbf{W}) = gS\partial_S\mathcal{H}(\mathbf{S})\cos\theta - c^2(\mathbf{W})\dfrac{A}{\mathbf{S}}$ and

$$c(\mathbf{W}) = \begin{cases} c & \text{for pressurised flow} \\ \sqrt{g\dfrac{A}{T(A)}\cos\theta} & \text{for free surface flow} \end{cases}$$

Integrating conservative PFS-System over $]X_{i-1/2}, X_{i+\frac{1}{2}}[\times[t_n, t_{n+1}[$, we can write a Finite Volume scheme as follows:

$$\begin{aligned} \mathbf{W}_i^{n+1} &= \mathbf{W}_i^n - \frac{\Delta t^n}{h_i}\left(\mathbf{F}(\mathbf{W}^*_{i+1/2}(0^-, \mathbf{W}_i^n, \mathbf{W}_{i+1}^n)) - \mathbf{F}(\mathbf{W}^*_{i-1/2}(0^+, \mathbf{W}_{i-1}^n, \mathbf{W}_i^n))\right) \\ &\quad + TS(\mathbf{W}_i^n) \end{aligned} \tag{5}$$

$\mathbf{W}^*_{i+1/2}(\xi = x/t, \mathbf{W}_i, \mathbf{W}_{i+1})$ is the exact or an approximate solution to the Riemann problem at interface $X_{i+1/2}$.

$W^*(0+, \mathbf{W}_i, \mathbf{W}_{i+1}) = (Z_{i+1}, \cos\theta_{i+1}, S_{i+1}, AP, QP)^t$ and
$W^*(0-, \mathbf{W}_i, \mathbf{W}_{i+1}) = (Z_{i+1}, \cos\theta_{i+1}, S_{i+1}, AM, QM)^t$ depend on two types of interfaces:

- a non transition point: the flow on both sides of the interface is of the same type
- a transition point: the flow changes of type through the interface

- approximating the convection matrix $D(\mathbf{W})$ by $\widetilde{D}$,

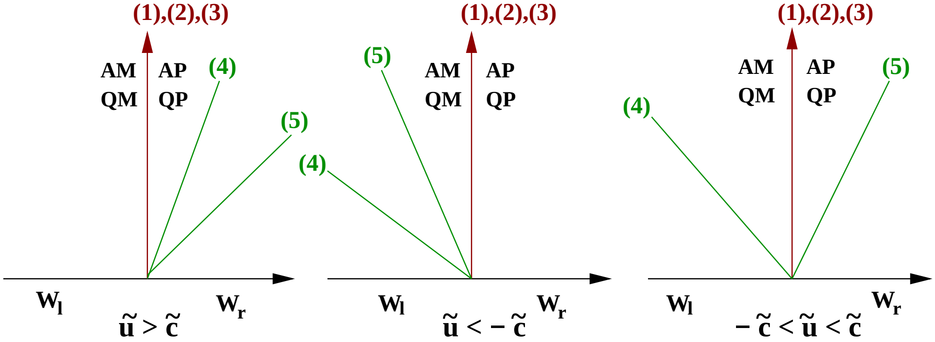
to compute (AM, QM), (AP, QP), we solve the linearized Riemann problem:

$$\left\{\begin{array}{lcl} \partial_t \mathbf{W} + \widetilde{D}\,\partial_X \mathbf{W} & = & 0 \\ \mathbf{W} & = & \left\{\begin{array}{lll} \mathbf{W}_l = (Z_l, \cos\theta_l, S_l, A_l, Q_l)^t & \text{if} & x < 0 \\ \mathbf{W}_r = (Z_r, \cos\theta_r, S_r, A_r, Q_r)^t & \text{if} & x > 0 \end{array}\right. \end{array}\right. \tag{5}$$

with $(\mathbf{W}_l, \mathbf{W}_r) = (\mathbf{W}_i, \mathbf{W}_{i+1})$ and $\widetilde{D} = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}})$ where $\widetilde{\mathbf{W}}$ is some approximate state of the left $\mathbf{W}_l$ and the right $\mathbf{W}_r$ state.

The eigenvalues of $\widetilde{D}$ are $\lambda_i = 0$, $i = 1, 2, 3$, $\lambda_4 = \widetilde{u} - c(\widetilde{\mathbf{W}})$, $\lambda_5 = \widetilde{u} + c(\widetilde{\mathbf{W}})$

where $c(\mathbf{W}) = \begin{cases} c & \text{for pressurised flow} \\ \sqrt{g\dfrac{A}{T(A)}\cos\theta} & \text{for free surface flow} \end{cases}$



We obtain, for instance in the subcritical case (when $-c(\widetilde{\mathbf{W}}) < \widetilde{u} < c(\widetilde{\mathbf{W}})$), we have:

$$AM = A_l + \frac{g\,\widetilde{A}}{2\,c(\widetilde{\mathbf{W}})\,(c(\widetilde{\mathbf{W}}) - \widetilde{u})}\,\psi_l^r + \frac{\widetilde{u} + c(\widetilde{\mathbf{W}})}{2\,c(\widetilde{\mathbf{W}})}\,(A_r - A_l) - \frac{1}{2\,c(\widetilde{\mathbf{W}})}\,(Q_r - Q_l)$$

$$QM = QP = Q_l - \frac{g\,\widetilde{A}}{2\,c(\widetilde{\mathbf{W}})}\,\psi_l^r + \frac{\widetilde{u}^2 - c(\widetilde{\mathbf{W}})^2}{2\,c(\widetilde{\mathbf{W}})}\,(A_r - A_l) - \frac{\widetilde{u} - c(\widetilde{\mathbf{W}})}{2\,c(\widetilde{\mathbf{W}})}\,(Q_r - Q_l)$$

$$AP = AM + \frac{g\,\widetilde{A}}{\widetilde{u}^2 - c(\widetilde{\mathbf{W}})^2}\,\psi_l^r$$

where ψ_l^r is the upwinded source term
$Z_r - Z_l + \mathcal{H}(\widetilde{\mathbf{S}})(\cos\theta_r - \cos\theta_l) + \Psi(\widetilde{\mathbf{W}})(S_r - S_l)$.

- Assumption the propagation of the interface (pressurized-free surface or free surface-pressurized) has a constant speed w during a time step.
- Consequently the half line $x = w\,t$ is the discontinuity line of $\widetilde{D}(W_l, W_r)$.
- Setting $w = \dfrac{Q^+ - Q^-}{A^+ - A^-}$ with $\mathbf{U}^- = (A^-, Q^-)$ and $\mathbf{U}^+ = (A^+, Q^+)$ the (unknown) states resp. on the left and on the right hand side of the line $x = w\,t$ click.
- Remark Both states $\mathbf{U}_l$ and $\mathbf{U}^-$ (resp. $\mathbf{U}_r$ and $\mathbf{U}^+$) correspond to the same type of flow
- Thus it makes sense to define the averaged matrices in each zone as follows:
  - for $x < w\,t$, we set $\widetilde{D}_l = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_l)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}_l$ and $\mathbf{W}^-$.
  - for $x > w\,t$, we set $\widetilde{D}_r = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_r)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}^+$ and $\mathbf{W}_r$.

- Assumption the propagation of the interface (pressurized-free surface or free surface-pressurized) has a constant speed w during a time step.
- Consequently the half line $x = w\,t$ is the discontinuity line of $\widetilde{D}(W_l, W_r)$.
- Setting $w = \dfrac{Q^+ - Q^-}{A^+ - A^-}$ with $\mathbf{U}^- = (A^-, Q^-)$ and $\mathbf{U}^+ = (A^+, Q^+)$ the (unknown) states resp. on the left and on the right hand side of the line $x = w\,t$ click.
- Remark Both states $\mathbf{U}_l$ and $\mathbf{U}^-$ (resp. $\mathbf{U}_r$ and $\mathbf{U}^+$) correspond to the same type of flow
- Thus it makes sense to define the averaged matrices in each zone as follows:
  - for $x < w\,t$, we set $\widetilde{D}_l = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_l)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}_l$ and $\mathbf{W}^-$.
  - for $x > w\,t$, we set $\widetilde{D}_r = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_r)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}^+$ and $\mathbf{W}_r$.

- Assumption the propagation of the interface (pressurized-free surface or free surface-pressurized) has a constant speed w during a time step.
- Consequently the half line $x = w\,t$ is the discontinuity line of $\widetilde{D}(W_l, W_r)$.
- Setting $w = \dfrac{Q^+ - Q^-}{A^+ - A^-}$ with $\mathbf{U}^- = (A^-, Q^-)$ and $\mathbf{U}^+ = (A^+, Q^+)$ the (unknown) states resp. on the left and on the right hand side of the line $x = w\,t$ click.
- Remark Both states $\mathbf{U}_l$ and $\mathbf{U}^-$ (resp. $\mathbf{U}_r$ and $\mathbf{U}^+$) correspond to the same type of flow
- Thus it makes sense to define the averaged matrices in each zone as follows:
  - for $x < w\,t$, we set $\widetilde{D}_l = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_l)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}_l$ and $\mathbf{W}^-$.
  - for $x > w\,t$, we set $\widetilde{D}_r = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_r)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}^+$ and $\mathbf{W}_r$.

- Assumption the propagation of the interface (pressurized-free surface or free surface-pressurized) has a constant speed w during a time step.
- Consequently the half line $x = w\,t$ is the discontinuity line of $\widetilde{D}(W_l, W_r)$.
- Setting $w = \dfrac{Q^+ - Q^-}{A^+ - A^-}$ with $\mathbf{U}^- = (A^-, Q^-)$ and $\mathbf{U}^+ = (A^+, Q^+)$ the (unknown) states resp. on the left and on the right hand side of the line $x = w\,t$ click.
- Remark Both states $\mathbf{U}_l$ and $\mathbf{U}^-$ (resp. $\mathbf{U}_r$ and $\mathbf{U}^+$) correspond to the same type of flow
- Thus it makes sense to define the averaged matrices in each zone as follows:
  - for $x < w\,t$, we set $\widetilde{D}_l = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_l)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}_l$ and $\mathbf{W}^-$.
  - for $x > w\,t$, we set $\widetilde{D}_r = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_r)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}^+$ and $\mathbf{W}_r$.

- Assumption the propagation of the interface (pressurized-free surface or free surface-pressurized) has a constant speed w during a time step.
- Consequently the half line $x = w\,t$ is the discontinuity line of $\widetilde{D}(W_l, W_r)$.
- Setting $w = \dfrac{Q^+ - Q^-}{A^+ - A^-}$ with $\mathbf{U}^- = (A^-, Q^-)$ and $\mathbf{U}^+ = (A^+, Q^+)$ the (unknown) states resp. on the left and on the right hand side of the line $x = w\,t$ click.
- Remark Both states $\mathbf{U}_l$ and $\mathbf{U}^-$ (resp. $\mathbf{U}_r$ and $\mathbf{U}^+$) correspond to the same type of flow
- Thus it makes sense to define the averaged matrices in each zone as follows:
  - for $x < w\,t$, we set $\widetilde{D}_l = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_l)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}_l$ and $\mathbf{W}^-$.
  - for $x > w\,t$, we set $\widetilde{D}_r = \widetilde{D}(\mathbf{W}_l, \mathbf{W}_r) = D(\widetilde{\mathbf{W}}_r)$ for some approximation $\widetilde{\mathbf{W}}_l$ which connects the state $\mathbf{W}^+$ and $\mathbf{W}_r$.

Then we formally solve two Riemann problems and use the Rankine-Hugoniot jump conditions through the line $x = w\,t$ which writes:

$$\begin{aligned} Q^+ - Q^- &= w\,(A^+ - A^-) \quad (5)\\ F_5(A^+, Q^+) - F_5(A^-, Q^-) &= w\,(Q^+ - Q^-) \quad (6) \end{aligned}$$

with $F_5(A, Q) = \dfrac{Q^2}{A} + p(X, A)$. According to $(\mathbf{U}^-, \mathbf{UM})$ and $(\mathbf{U}^+, \mathbf{UP})$ (unknowns) at the interface $x_{i+1/2}$ and the sign of the speed w, we have to deal with four cases:

- pressure state propagating downstream click,
- pressure state propagating upstream,
- free surface state propagating downstream,
- free surface state propagating upstream.

Then we formally solve two Riemann problems and use the Rankine-Hugoniot jump conditions through the line $x = w\,t$ which writes:

$$\begin{aligned} Q^+ - Q^- &= w\,(A^+ - A^-) && (5) \\ F_5(A^+, Q^+) - F_5(A^-, Q^-) &= w\,(Q^+ - Q^-) && (6) \end{aligned}$$

with $F_5(A, Q) = \dfrac{Q^2}{A} + p(X, A)$. According to $(\mathbf{U}^-, \mathbf{UM})$ and $(\mathbf{U}^+, \mathbf{UP})$ (unknowns) at the interface $x_{i+1/2}$ and the sign of the speed w, we have to deal with four cases:

- pressure state propagating downstream click,
- pressure state propagating upstream,
- free surface state propagating downstream,
- free surface state propagating upstream.

Outline

1. Modelisation: the pressurized and free surface flows model
   - Previous works about mixed model
   - The free surface model
   - The pressurized model
   - The PFS-model : a natural coupling
2. Finite Volume discretisation
   - Discretisation of the space domain
   - Explicit first order VFRoe scheme
     1. The Case of a non transition point
     2. The Case of a transition point
     3. Update of the cell state
     4. Approximation of the convection matrix
3. Numerical experiments
4. Conclusion and perspectives

State update

Given n, $\forall i$, A_i^n and E_i^n are known. Then

- if $E_i^n = 0$ then

  if $A_i^{n+1} < S_i$ then $E_i^{n+1} = 0$
  else $E_i^{n+1} = 1$

- if $E_i^n = 1$ then

  if $A_i^{n+1} \geqslant S_i$ then $E_i^{n+1} = 1$
  else $E_i^{n+1} = E_{i-1}^n E_{i+1}^n$

State update

Given n, $\forall i$, A_i^n and E_i^n are known. Then

- if $E_i^n = 0$ then

  if $A_i^{n+1} < S_i$ then $E_i^{n+1} = 0$
  else $E_i^{n+1} = 1$

- if $E_i^n = 1$ then

  if $A_i^{n+1} \geqslant S_i$ then $E_i^{n+1} = 1$
  else $E_i^{n+1} = E_{i-1}^n E_{i+1}^n$

Given n, $\forall i$, A_i^n and E_i^n are known. Then

- if $E_i^n = 0$ then

  if $A_i^{n+1} < S_i$ then $E_i^{n+1} = 0$
  else $E_i^{n+1} = 1$

- if $E_i^n = 1$ then

  if $A_i^{n+1} \geqslant S_i$ then $E_i^{n+1} = 1$
  else $E_i^{n+1} = E_{i-1}^n E_{i+1}^n$

Given n, $\forall i$, A_i^n and E_i^n are known. Then

- if $E_i^n = 0$ then

  if $A_i^{n+1} < S_i$ then $E_i^{n+1} = 0$
  else $E_i^{n+1} = 1$

- if $E_i^n = 1$ then

  if $A_i^{n+1} \geqslant S_i$ then $E_i^{n+1} = 1$
  else $E_i^{n+1} = E_{i-1}^n E_{i+1}^n$

Given n, $\forall i$, A_i^n and E_i^n are known. Then

- if $E_i^n = 0$ then

  if $A_i^{n+1} < S_i$ then $E_i^{n+1} = 0$
  else $E_i^{n+1} = 1$

- if $E_i^n = 1$ then

  if $A_i^{n+1} \geqslant S_i$ then $E_i^{n+1} = 1$
  else $E_i^{n+1} = E_{i-1}^n E_{i+1}^n$

Given n, $\forall i$, A_i^n and E_i^n are known. Then

- if $E_i^n = 0$ then

  if $A_i^{n+1} < S_i$ then $E_i^{n+1} = 0$
  else $E_i^{n+1} = 1$

- if $E_i^n = 1$ then

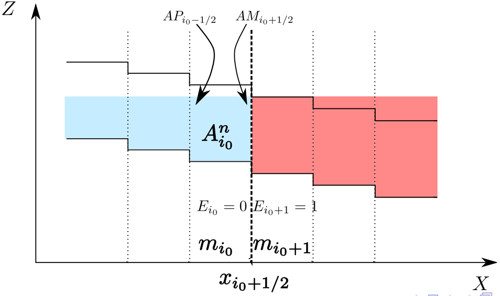
  if $A_i^{n+1} \geqslant S_i$ then $E_i^{n+1} = 1$
  else $E_i^{n+1} = E_{i-1}^n E_{i+1}^n$

The classical approximation $D(\widetilde{\mathbf{W}})$ of the Roe matrix
$D_{Roe}(\mathbf{W}_l, \mathbf{W}_r) = \displaystyle\int_0^1 D(\mathbf{W}_r + (1-s)(\mathbf{W}_l - \mathbf{W}_r))\, ds$ defined by
$\widetilde{D} = D(\widetilde{\mathbf{W}}) = D\left(\dfrac{\mathbf{W}_l + \mathbf{W}_r}{2}\right)$ preserve the still water steady state only for constant section pipe and $Z = 0$.

Let us start with the consideration: the still water steady state is perfectly maintained: it exists n such that for every i, if $Q_i^n = 0$ and $\forall i$,

$$\begin{aligned}
\text{A1:}\quad & c^2 \ln\left(\frac{A_{i+1}^n}{S_{i+1}}\right) + g\mathcal{H}(\mathbf{S}_{i+1}^n)\cos\theta + gZ_{i+1} = \\
& c^2 \ln\left(\frac{A_i^n}{S_i}\right) + g\mathcal{H}(\mathbf{S}_i^n)\cos\theta + gZ_i, \\
\text{A2:}\quad & AM_{i+1/2}^n = AP_{i-1/2}^n, \\
\text{A3:}\quad & Q_{i+1/2}^n = Q_{i-1/2}^n,
\end{aligned}$$

then, for all $l > n$ the conditions A1, A2 and A3 holds.



$(\widetilde{A}^n_{i-1/2}, \widetilde{A}^n_{i+1/2})$ as the solution of the non-linear system:

$$\left\{ \begin{array}{lcl} 0 & = & \Delta A^n_{i+1/2} + \dfrac{g}{2}\left(\dfrac{\widetilde{A}^n_{i+1/2}\psi_i^{i+1}}{\widetilde{c}^2_{i+1/2}} + \dfrac{\widetilde{A}^n_{i-1/2}\psi_{i-1}^{i}}{\widetilde{c}^2_{i-1/2}} \right) \\ 0 & = & \dfrac{g}{2}\left\{ \dfrac{\widetilde{A}^n_{i-1/2}\,\psi_{i-1}^{i}}{\widetilde{c}_{i-1/2}} - \dfrac{\widetilde{A}^n_{i+1/2}\,\psi_i^{i+1}}{\widetilde{c}_{i+1/2}} \right\} + \dfrac{\Delta A^n_{i+1/2}}{2}\left(\widetilde{c}_{i-1/2} - \widetilde{c}_{i+1/2}\right) \end{array} \right. \quad (7)$$

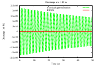
the numerical scheme is exactly well-balanced.

For small Δx, we show that

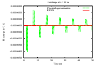
$$\widetilde{A}^n_{i+1/2} \approx \frac{A^n_i + A^n_{i+1}}{2}$$

Outline

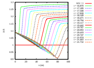
- Well-balanced scheme and the averaged approximation for P



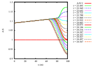
- Well-balanced scheme and the averaged approximation for FS



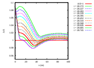
- Depression for a contracting pipe



- Depression for an uniform pipe



- Depression for an expanding pipe



1. Modelisation: the pressurized and free surface flows model
   - Previous works about mixed model
   - The free surface model
   - The pressurized model
   - The PFS-model : a natural coupling

2. Finite Volume discretisation
   - Discretisation of the space domain
   - Explicit first order VFRoe scheme
     1. The Case of a non transition point
     2. The Case of a transition point
     3. Update of the cell state
     4. Approximation of the convection matrix

3. Numerical experiments

4. Conclusion and perspectives

Conclusion

- Easy implementation of source terms
- Very good agreement for uniform case
- Still water steady states are preserved

Perspective

- Air entrainment treated as a bilayer fluid flow (in progress).
- Diphasic approach to take into account air entrapment, evaporation/condensation and cavitation.

Conclusion

- Easy implementation of source terms
- Very good agreement for uniform case
- Still water steady states are preserved

Perspective

- Air entrainment treated as a bilayer fluid flow (in progress).
- Diphasic approach to take into account air entrapment, evaporation/condensation and cavitation.

Thank you for your attention