Air entrainment in transient flows in closed water pipes: A two-layer approach.

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joint work with C. Bourdarias and S. Gerbi, LAMA, Chambéry, France

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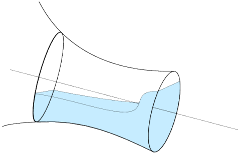
Outline of the talk

1. Physical and mathematical motivations
   - Air entrainement
   - Previous works
2. The two layer model
   - Fluid Layer : incompressible Euler's Equations
   - Air Layer : compressible Euler's Equations
   - The two-layer model
   - Properties
3. Numerical approximation
   - The kinetic scheme
   - A numerical experiment

OUTLINE

The air entrainment

- appears in the transient flow in closed pipes not completely filled : the liquid flow (as well as the air flow) is free surface.



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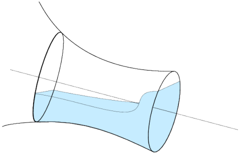
- may lead to two-phase flows for transition : free surface flows → pressurized flows.



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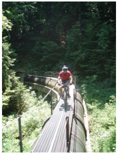
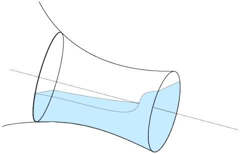
(f) Sewers . . . in Paris



(g) Forced pipe

The air entrainment

- appears in the transient flow in closed pipes not completely filled : the liquid flow (as well as the air flow) is free surface.
- may lead to two-phase flows for transition : free surface flows $\rightarrow$ pressurized flows.
- may cause severe damage due to the pressure surge.



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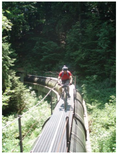

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(l) . . . at Minnesota http://www.sewerhistory.org/grfx/misc/disaster.htm

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- the two layer model (Ersoy *et al* M2AN 2013).

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- $\Rightarrow$ Kelvin-Helmholtz instability, for which the two-layer model is not a priori suitable :

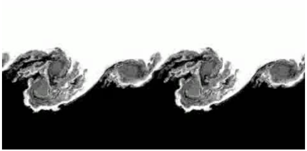

Figure: Kelvin-Helmholtz instability (source : wiki Kelvin-Helmholtz instability)

Outline

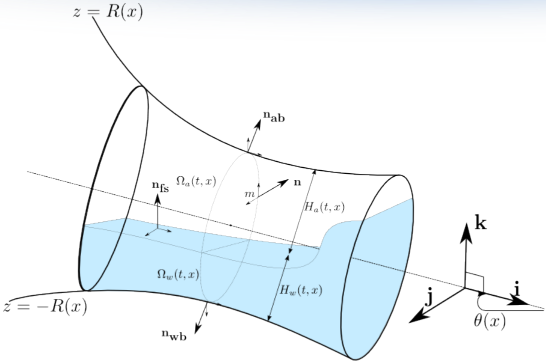

FIGURE: Geometric characteristics of the domain.

We have then the first natural coupling :

$$H_w(t,x) + H_a(t,x) = 2R(x)\,.$$

Outline

Incompressible Euler's equations

$$\begin{array}{lcll} \mathrm{div}(\rho_0 \mathbf{U_w}) & = & 0, & \text{on } \mathbb{R} \times \Omega_{t,w} \\ \partial_t(\rho_0 \mathbf{U_w}) + \mathrm{div}(\rho_0 \mathbf{U_w} \otimes \mathbf{U_w}) + \nabla P_w & = & \rho_0 \mathbf{F}, & \text{on } \mathbb{R} \times \Omega_{t,w} \end{array}$$

where $\mathbf{U_w}(t,x,y,z) = (U_w, V_w, W_w)$ the velocity, $P_w(t,x,y,z)$ the pressure, $\mathbf{F}$ the gravity strength.

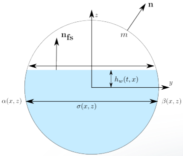

Figure: Cross-section of the domain

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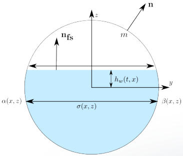
- Write non dimensional form of Euler equations using the parameter $\epsilon = H/L \ll 1$ and takes $\epsilon = 0$.
- Equality of the pressure of air and water $P_a = P_w$ at the free surface interface.
- Section averaging $\overline{\rho U} \approx \overline{\rho}\overline{U}$ and $\overline{U^2} \approx \overline{U}\,\overline{U}$.
- Introduce $A(t,x) = \displaystyle\int_{\Omega_w} dydz$, $u(t,x) = \dfrac{1}{A(t,x)} \displaystyle\int_{\Omega_w} U_w(t,x,y,z)\, dydz$, and $Q(t,x) = A(t,x)u(t,x)$.



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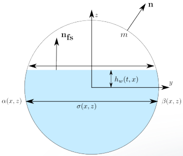

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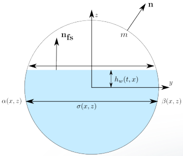

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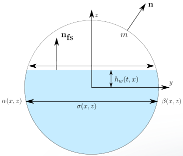

Figure: Cross-section of the domain

Fluid layer model

$$\left\{\begin{array}{lcl}\partial_t A + \partial_x Q & = & 0 \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + A P_a(\overline{\rho})/\rho_0 + g I_1(x,A)\cos\theta\right) & = & -gA\partial_x Z \\ & & +g I_2(x,A)\cos\theta \\ & & +P_a(\overline{\rho})/\rho_0\,\partial_x A \end{array}\right.$$

where

the hydrostatic pressure	:	$I_1(x,A) = \displaystyle\int_{-R}^{h_w} (h_w - z)\sigma(x,z)\,dz,$
the pressure source term	:	$I_2(x,A) = \displaystyle\int_{-R}^{h_w} (h_w - z)\partial_x\sigma(x,z)\,dz\,,$
the air pressure	:	$P_a.$

1. Physical and mathematical motivations
   - Air entrainement
   - Previous works
2. The two layer model
   - Fluid Layer : incompressible Euler's Equations
   - **Air Layer : compressible Euler's Equations**
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COMPRESSIBLE EULER'S EQUATIONS

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where $\mathbf{U_a}(t,x,y,z) = (U_a, V_a, W_a)$ the velocity, $P_a(t,x,y,z)$ the pressure, $\rho_a(t,x,y,z)$ the density.

Compressible Euler's equations

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with

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- Write non dimensional form of Euler equations using the parameter $\epsilon = H/L \ll 1$ and takes $\epsilon = 0$.
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- Section averaging Averaged nonlinearity $\sim$ Nonlinearity of the averaged.
- Introduce $\mathcal{A}(t,x) = \displaystyle\int_{\Omega_a} dydz$, $uv(t,x) = \dfrac{1}{\mathcal{A}(t,x)} \displaystyle\int_{\Omega_a} U_a(t,x,y,z)\,dydz$, $M = \overline{\rho}/\rho_0\mathcal{A}$, $D = Mv$ and $c_a^2 = \dfrac{\partial p}{\partial \rho} = k\gamma \left(\dfrac{\rho_0 M}{\mathcal{A}}\right)^{\gamma-1}$.

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Air layer model

$$\left\{\begin{array}{rcl}\partial_t M + \partial_x D & = & 0\\ \partial_t D + \partial_x\left(\dfrac{D^2}{M} + \dfrac{M}{\gamma}c_a^2\right) & = & \dfrac{M}{\gamma}c_a^2\,\partial_x(\mathcal{A})\end{array}\right. .$$

where

the γ pressure : $\dfrac{M}{\gamma}c_a^2$,

the pressure source term : $\dfrac{M}{\gamma}c_a^2\,\partial_x(\mathcal{A})$.

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$\mathcal{A} + A = S$ where $S = S(x)$ denotes the pipe section

Two-layer model

$$
\left\{
\begin{array}{rcl}
\partial_t M + \partial_x D & = & 0 \\
\partial_t D + \partial_x \left(\dfrac{D^2}{M} + \dfrac{M}{\gamma} c_a^2 \right) & = & \dfrac{M}{\gamma} c_a^2 \, \partial_x (S - A) \\
\\
\partial_t A + \partial_x Q & = & 0 \\
\partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(x, A) \cos\theta + \dfrac{A}{(S-A)} \dfrac{M}{\gamma} c_a^2 \right) & = & -gA\partial_x Z \\
 & & +g I_2(x, A) \cos\theta \\
 & & + \dfrac{A}{(S-A)} \dfrac{M}{\gamma} c_a^2 \, \partial_x A
\end{array}
\right. .
$$

OUTLINE

1. Energies

$$E_a = \frac{Mv^2}{2} + \frac{c_a^2\, M}{\gamma(\gamma-1)} \text{ and } E_w = \frac{Au^2}{2} + gA(h_w - I_1(x,A)/A)\cos\theta + gAZ$$

satisfy the following entropy flux equalities :

$$\partial_t E_a + \partial_x H_a = \frac{c_a^2\, M}{\gamma(S-A)}\,\partial_t A$$

and

$$\partial_t E_w + \partial_x H_w = -\frac{c_a^2\, M}{\gamma(S-A)}\,\partial_t A$$

where

$$H_a = \left(E_a + \frac{c_a^2\, M}{\gamma}\right) v \text{ and } H_w = \left(E_w + gI_1(x,A)\cos\theta + A\frac{c_a^2\, M}{(S-A)}\right) u\ .$$

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where

$$H_a = \left(E_a + \frac{c_a^2\, M}{\gamma}\right) v \text{ and } H_w = \left(E_w + gI_1(x,A)\cos\theta + A\frac{c_a^2\, M}{(S-A)}\right) u\ .$$

2. The total energy satisfies

$$\partial_t \mathcal{E} + \partial_x \mathcal{H} = 0\ .$$

- Quasi-linear form : $\mathbf{W} = (M, D, A, Q)^t$

$$\partial_t \mathbf{W} + \mathcal{D}(x, \mathbf{W}) \partial_X \mathbf{W} = 0$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ c_a^2 - v^2 & 2v & \dfrac{M}{S-A} c_a^2 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{A}{(S-A)} c_a^2 & 0 & c_w^2 + \dfrac{AM}{(S-A)^2} c_a^2 - u^2 & 2u \end{pmatrix}$$

where $c_m := c_w^2 + \dfrac{AM}{(S-A)^2} c_a^2$: water sound speed under the air effect.

- Quasi-linear form : $\mathbf{W} = (M, D, A, Q)^t$

$$\partial_t \mathbf{W} + \mathcal{D}(x, \mathbf{W}) \partial_X \mathbf{W} = 0$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ c_a^2 - v^2 & 2v & \dfrac{M}{S-A} c_a^2 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{A}{(S-A)} c_a^2 & 0 & c_w^2 + \dfrac{AM}{(S-A)^2} c_a^2 - u^2 & 2u \end{pmatrix}$$

where $c_m := c_w^2 + \dfrac{AM}{(S-A)^2} c_a^2$: water sound speed under the air effect.

- Writing

$$F = \frac{v-u}{c_m}, \quad \sqrt{H} = \frac{c_a}{c_m}, \quad c_m = \sqrt{c_w^2 + s c_a^2} \text{ with } s = \frac{AM}{(S-A)^2} \geqslant 0\,,$$

the characteristic polynom reads $P(x = \lambda / cm) =$

$$x^4 - 2(2+F)x^3 + ((1+F)(5+F) - H)\, x^2 + 2\left(H - (1+F)^2\right) x - sH^2$$

where λ stands for an eigenvalue of $\mathcal{D}$.

Theorem (Fuller, IEEE Trans. Automat. Control, 81)

All the root of Equation P are real if and only if one of the following conditions holds :

(i) $\Delta_3 > 0$, $\Delta_5 > 0$ *and* $\Delta_7 \geqslant 0$,

(ii) $\Delta_3 \geqslant 0$, $\Delta_5 = 0$ *and* $\Delta_7 = 0$

where Δ_3, Δ_5, Δ_7 *are the inner determinant of the discriminant of* P.

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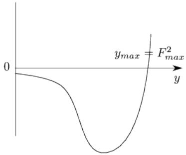
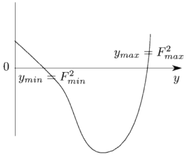


(a) $y_{min} < 0$



(b) $y_{min} > 0$

Figure: Behavior of the polynomial $\Delta_7(y)$

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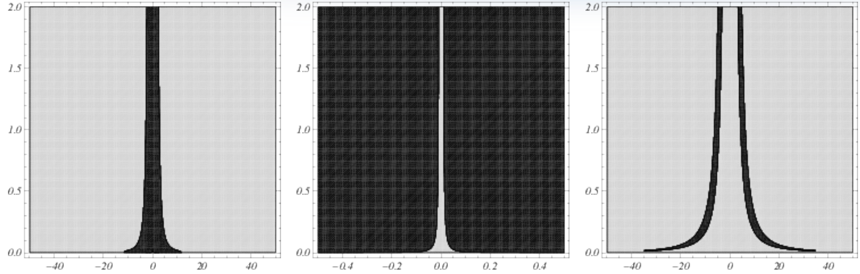
THEOREM (ERSOY *et al.*, M2AN,13)

The two-layer system is strictly hyperbolic if

- $y = F^2 \geqslant F^2_{max} := y^2_{max} \Longleftrightarrow$ *large relative speed*
- $y_{min} > 0$ *and* $0 \leqslant F^2 \leqslant y_{min} = F^2_{min} \Longleftrightarrow$ *small relative speed*

where $s = \dfrac{\rho}{\rho_0}\dfrac{A}{S-A} \sim \dfrac{\rho_2}{\rho_1} < 1$ *as in the two-layer shallows water equations.*



(a) large relative speed ($\rho = 1000$ (air density), F x-axis and A y-axis)

(b) Small relative speed(Zoom on :$\rho = 1000$, F x-axis and A y-axis)

(c) $\rho = 10$, F x-axis and A y-axis

Figure: black grey = non hyperbolic region

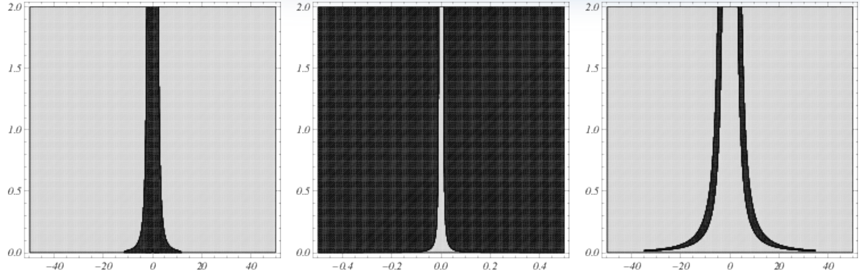


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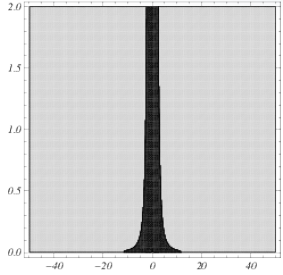
FIGURE: black grey = non hyperbolic region

As a consequence

- system may lose its hyperbolicity.
- solver based on the computation of eigenvalues are useless.



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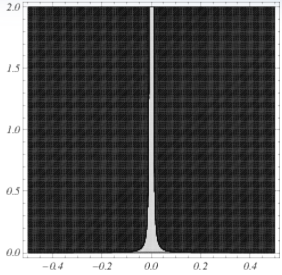

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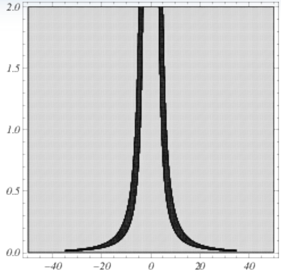

(c) $\rho = 10$, F x-axis and A y-axis

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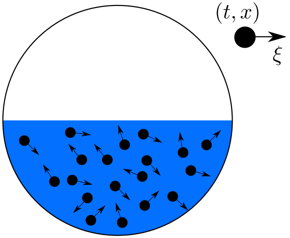
As in gas theory,
Describe the macroscopic behavior from particle motions, here, assumed fictitious by introducing

$$\left\{ \begin{array}{l} \text{a } \chi \text{ density function and} \\ \text{a } \mathcal{M}(t,x,\xi;\chi) \text{ maxwellian function (or a Gibbs equilibrium)} \end{array} \right.$$



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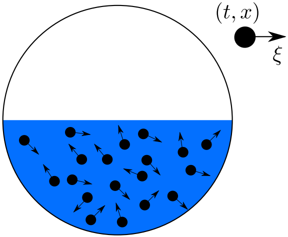

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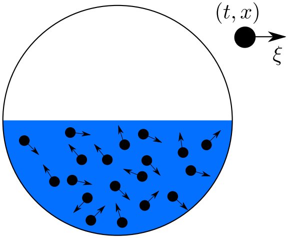

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Thus, to be able to define the numerical *macroscopic fluxes* from the microscopic one.

...Faire d'une pierre deux coups...

We introduce

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MICRO-MACROSCOPIC RELATIONS

$$\begin{aligned} A &= \int_{\mathbb{R}} \mathcal{M}(t,x,\xi) \, d\xi \\ Au &= \int_{\mathbb{R}} \xi \mathcal{M}(t,x,\xi) \, d\xi \\ Au^2 + Ab^2 &= \int_{\mathbb{R}} \xi^2 \mathcal{M}(t,x,\xi) \, d\xi \end{aligned}$$

THE KINETIC FORMULATION [PERTHAME, OXFORD LECT. SER. IN MATH. AND ITS APPLIC., 02]

(A, Q) is solution of the (air or water) system if and only if $\mathcal{M}$ satisfies the transport equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\Phi\, \partial_\xi \mathcal{M} = \mathcal{K}(t, x, \xi)$$

where $\mathcal{K}(t, x, \xi)$ is a collision kernel satisfying a.e. (t, x)

$$\int_{\mathbb{R}} \mathcal{K}\, d\xi = 0\ , \ \int_{\mathbb{R}} \xi\, \mathcal{K} d\, \xi = 0$$

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The kinetic formulation [Perthame, Oxford Lect. Ser. in Math. and its Applic., 02]

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General form of the source terms :

$$\Phi = \overbrace{\frac{d}{dx} Z}^{\text{conservative}} + \overbrace{\mathbf{B} \cdot \frac{d}{dx} \mathbf{W}}^{\text{non conservative}} + \overbrace{K \frac{Q|Q|}{A^2}}^{\text{friction}}$$

- conservative term : classical upwind
- non conservative term : mid point rule (DLM, 95)
- friction : dynamic topography (Ersoy, Ph.D.)

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$$\left\{\begin{array}{lcl} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}(t_n, x, \xi) \stackrel{def}{:=} \sqrt{h(t,x)}\chi\left(\dfrac{\xi - u(t_n, x, \xi)}{\sqrt{h(t,x)}}\right) \end{array}\right.$$

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- i.e.

$$f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \xi \frac{\Delta t^n}{\Delta x} \left(\mathcal{M}^-_{i+\frac{1}{2}}(\xi) - \mathcal{M}^+_{i-\frac{1}{2}}(\xi)\right)$$

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- i.e.

$$\mathbf{U}_i^{n+1} = \begin{pmatrix} A_i^{n+1} \\ Q_i^{n+1} \end{pmatrix} \overset{def}{:=} \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f_i^{n+1}(\xi) \, d\xi$$

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- i.e.

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x}\left(F^-_{i+1/2} - F^+_{i-1/2}\right) \text{ with } F^\pm_{i\pm\frac{1}{2}} = \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}^\pm_{i\pm\frac{1}{2}}(\xi)\, d\xi.$$

OUTLINE

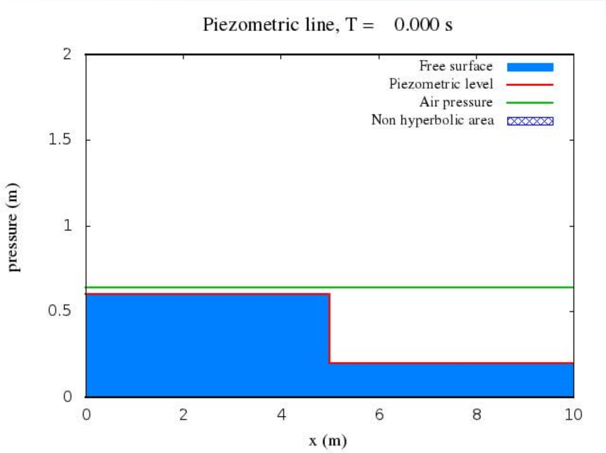

Figure: "Dam break in presence of air in a closed water pipe."

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A lot of things . . . to do

- air entrapment and mixed flows
- more realistic models based on

Thank you

for your

attention

$$\Delta_3 = \begin{pmatrix} a_0 & a_1 & a_2 \\ 0 & 4a_0 & 3a_1 \\ 4a_0 & 3a_1 & 2a_2 \end{pmatrix}, \quad \Delta_5 = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \end{pmatrix},$$

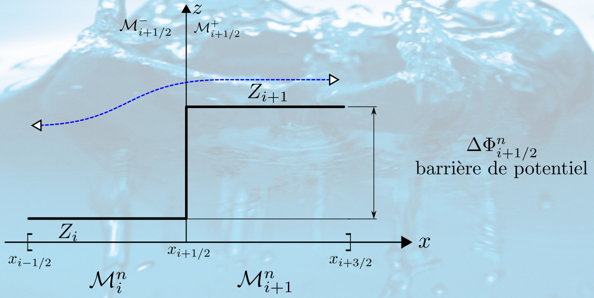
$$\Delta_7 = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \end{pmatrix}.$$

THEOREM

For smooth solutions, the velocities u and v satisfy

$$\begin{aligned}
\partial_t v + \partial_x \left(\frac{v^2}{2} + \frac{c_a^2}{\gamma - 1} \right) &= 0 \, , \\
\partial_t u + \partial_x \left(\frac{u^2}{2} + g h_w(A) \cos\theta + gZ + \frac{c_a^2 \, M}{\gamma(S - A)} \right) &= 0 \, .
\end{aligned}$$

$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}_i^n(\xi)}^{\text{positive transmission}} + \underbrace{\mathbb{1}_{\{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}>0\}}\mathcal{M}^n_{i+1}\left(-\sqrt{\xi^2-2g\Delta\Phi^n_{i+1/2}}\right)}_{\text{negative transmission}}$$



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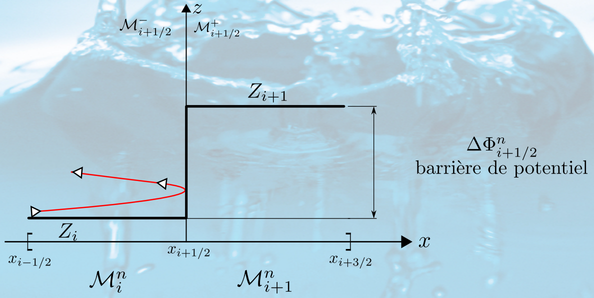

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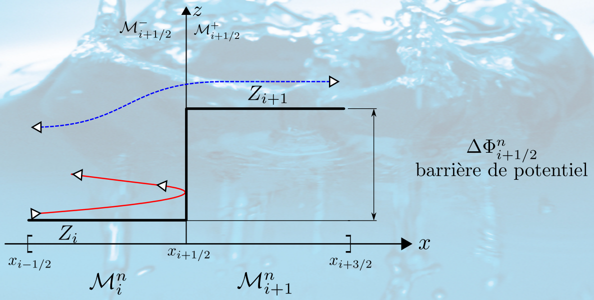

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