basque center for applied mathematics

A Well Balanced Finite Volume Kinetic (FVK) scheme for unsteady mixed flows in non uniform closed water pipes.

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Laboratoire de Mathématiques Jean Leray, Nantes, the 24 March 2011

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1. # Unsteady mixed flows : PFS equations (Pressurized and Free Surface)
   - Previous works
   - Formal derivation of the free surface and pressurized model
   - A coupling : the PFS-model

2. # A Finite Volume Framework
   - Kinetic Formulation and numerical scheme
   - The χ function and well balanced scheme
     1. Classical scheme fails in presence of complex source terms
     2. An alternative toward a Well-Balanced scheme
   - Numerical results

3. # Conclusion and perspectives

OUTLINE

1. UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)
   - Previous works
   - Formal derivation of the free surface and pressurized model
   - A coupling : the PFS-model
2. A FINITE VOLUME FRAMEWORK
   - Kinetic Formulation and numerical scheme
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     1. Classical scheme fails in presence of complex source terms
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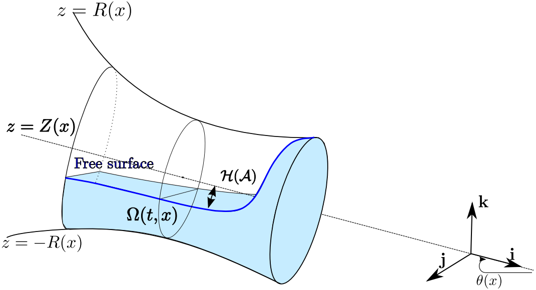
3. CONCLUSION AND PERSPECTIVES

Unsteady mixed flows in closed water pipes ?

- Free surface area (SL)
  sections are not completely filled and the flow is incompressible...



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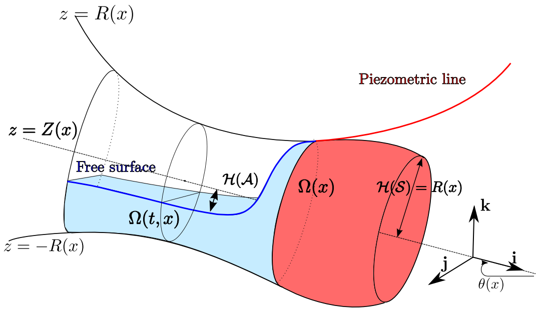
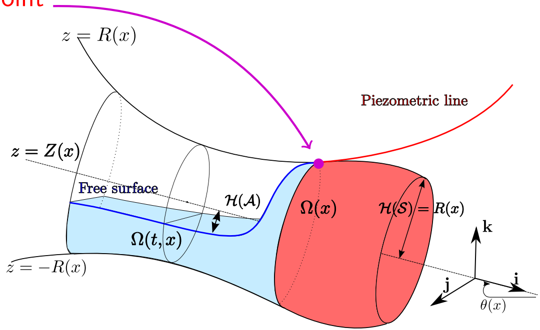
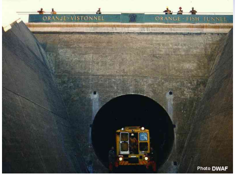


Unsteady mixed flows in closed water pipes ?

- Free surface area (SL)
  sections are not completely filled and the flow is incompressible. . .
- Pressurized area (CH)
  sections are non completely filled and the flow is compressible. . .
- Transition point



Examples of pipes



Orange-Fish tunnel

Sewers . . . in Paris

Forced pipe

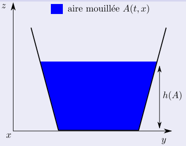
problems . . . at Minnesota
http://www.sewerhistory.org/grfx/misc/disaster.htm

Outline

Generally

Saint-Venant equations :

$$\left\{ \begin{array}{l} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(A) \right) = 0 \end{array} \right.$$



with

$A(t,x)$	:	wet area
$Q(t,x)$	:	discharge
$I_1(A)$	:	hydrostatic pressure
g	:	gravity

Advantage

- Conservative formulation $\longrightarrow$ Easy numerical implementation

Hamam and McCorquodale (82), Trieu Dong (91), Musandji Fuamba (02), Vasconcelos *et al* (06)

Previous works

For pressurized flows :

Generally

Allievi equations :

$$\left\{\begin{array}{l}\partial_t p + \dfrac{c^2}{gS}\partial_x Q = 0,\\ \partial_t Q + gS\partial_x p = 0\end{array}\right.$$

with
$p(t,x)$: pressure
$Q(t,x)$: discharge
$c(t,x)$: sound speed
$S(x)$: section

Advantage

- Compressibility of water is taking into account $\Longrightarrow$ Sub-atmospheric flows and over-pressurized flows are well computed

Drawback

- Non conservative formulation $\Longrightarrow$ Cannot be, at least easily, coupled to Saint-Venant equations

Winckler (93), Blommaert (00)

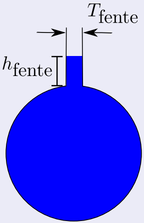
Previous works

For mixed flows :

Generally

Saint-Venant with Preissmann slot artifact :

$$\begin{cases} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(A) \right) = 0 \end{cases}$$



Advantage

- Only one model for two types of flows.

Drawbacks

- Incompressible Fluid $\Longrightarrow$ Water hammer not well computed
- Pressurized sound speed $\simeq \sqrt{S/T_{\text{fente}}} \Longrightarrow$ adjustment of T_{fente}
- Depression $\Longrightarrow$ seen as a free surface state

Preissmann (61), Cunge *et al.* (65), Baines *et al.* (92), Garcia-Navarro *et al.* (94), Capart *et al.* (97), Tseng (99)

Our goal :

- Use Saint-Venant equations for free surface flows

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- Write a pressurized model
  - which takes into account the compressibility of water
  - which takes into account the depression
  - similar to Saint-Venant equations

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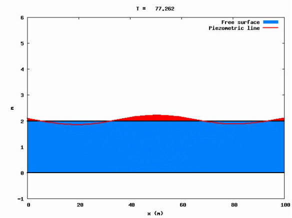
- Use Saint-Venant equations for free surface flows
- Write a pressurized model
  - which takes into account the compressibility of water
  - which takes into account the depression
  - similar to Saint-Venant equations
- Get one model for mixed flows

To be able to simulate, for instance :



C. Bourdarias and S. Gerbi
A finite volume scheme for a model coupling free surface and pressurized flows in pipes.
J. Comp. Appl. Math., 209(1) :109–131, 2007.

Derivation of the free surface model

$3D$ Incompressible Euler equations

$$\begin{array}{rcl} \rho_0 \mathrm{div}(\mathbf{U}) & = & 0 \\ \rho_0(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) + \nabla p & = & \rho_0 F \end{array}$$

Method :

1. Write Euler equations in curvilinear coordinates.
2. Write equations in non-dimensional form using the small parameter $\epsilon = H/L$ and takes $\epsilon = 0$.
3. Section averaging $\overline{U^2} \approx \overline{U}\,\overline{U}$ and $\overline{U\,V} \approx \overline{U}\,\overline{V}$.
4. Introduce $A_{sl}(t,x)$: wet area, $Q_{sl}(t,x)$ discharge given by :

$$A_{sl}(t,x) = \int_{\Omega(t,x)} dydz, \quad Q_{sl}(t,x) = A_{sl}(t,x)u(t,x)$$

$$u(t,x) = \frac{1}{A_{sl}(t,x)} \int_{\Omega(t,x)} U(t,x)\; dydz$$

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J.-F. Gerbeau, B. Perthame
Derivation of viscous Saint-Venant System for Laminar Shallow Water ; Numerical Validation.
Discrete and Continuous Dynamical Systems, Ser. B, Vol. 1, Num. 1, 89–102, 2001.

F. Marche
Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects.
European Journal of Mechanic B/Fluid, 26 (2007), 49–63.

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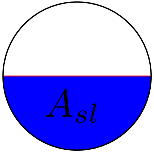

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The free surface model

$$\begin{cases} \partial_t A_{sl} + \partial_x Q_{sl} & = & 0, \\ \partial_t Q_{sl} + \partial_x \left(\dfrac{Q_{sl}^2}{A_{sl}} + p_{sl}(x, A_{sl}) \right) & = & -g A_{sl} \dfrac{d\,Z}{dx} + Pr_{sl}(x, A_{sl}) - G(x, A_{sl}) \end{cases}$$

with

$$p_{sl} = g I_1(x, A_{sl}) \cos\theta \quad : \text{hydrostatic pressure law}$$

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$$K = \frac{1}{K_s^2 R_h(A_{sl})^{4/3}} \quad : \text{Manning-Strickler law}$$

Derivation of the pressurized model

3D isentropic compressible equations

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{U}) = 0$$
$$\partial_t(\rho \mathbf{U}) + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{F}$$

with

$$p = p_a + \frac{\rho - \rho_0}{c^2} \text{ with } c \text{ sound speed}$$

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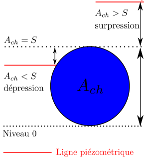

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Continuity criterion

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« mixed » condition

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To be coupled

The PFS model

The « mixed » variable

We introduce a state indicator

$$E = \begin{cases} 1 & \text{if the flow is pressurized (CH),} \\ 0 & \text{if the flow is free surface (SL)} \end{cases}$$

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and the physical section of water **S** by :

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We set

$$\begin{aligned} A &= \frac{\bar{\rho}}{\rho_0}\mathbf{S} = \begin{cases} \mathbf{S}(A_{sl}, 0) = A_{sl} & \text{if SL} \\ \frac{\bar{\rho}}{\rho_0}\mathbf{S}(A_{sl}, 1) = A_{ch} & \text{if CH} \end{cases} && : \text{ the « mixed »variable} \\ Q &= Au && : \text{ the discharge} \end{aligned}$$

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$$\mathbf{S} = \mathbf{S}(A_{sl}, E) = \begin{cases} S & \text{if} \quad E = 1, \\ A_{sl} & \text{if} \quad E = 0. \end{cases}$$

We set

$$\begin{array}{rcl} A & = & \dfrac{\bar{\rho}}{\rho_0}\mathbf{S} = \begin{cases} \mathbf{S}(A_{sl}, 0) = A_{sl} & \text{if SL} \\ \dfrac{\bar{\rho}}{\rho_0}\mathbf{S}(A_{sl}, 1) = A_{ch} & \text{if CH} \end{cases} \\ Q & = & Au \end{array}$$

A : the « mixed » variable

Q : the discharge

Continuity of S at transition point

- Continuity of $\mathbf{S} \Longrightarrow$ continuity of p at transition point

$\longrightarrow$

$$p(x, A, E) = c^2(A - \mathbf{S}) + gI_1(x, \mathbf{S})\cos\theta$$

- Continuity of **S** $\Longrightarrow$ continuity of p at transition point

  $\longrightarrow$

$$p(x, A, E) = c^2 (A - \mathbf{S}) + g I_1(x, \mathbf{S}) \cos\theta$$

- Similar construction for the pressure source term :

$$Pr(x, A, E) = c^2 \left(\frac{A}{\mathbf{S}} - 1 \right) \frac{d\,S}{dx} + g I_2(x, \mathbf{S}) \cos\theta$$

The PFS model

$$\begin{cases} \partial_t(A) + \partial_x(Q) & = 0 \\ \partial_t(Q) + \partial_x\left(\dfrac{Q^2}{A} + p(x,A,E)\right) & = -g\,A\,\dfrac{d}{dx}Z(x) \\ & +Pr(x,A,E) \\ & -G(x,A,E) \\ & -g\,K(x,\mathbf{S})\,\dfrac{Q|Q|}{A} \end{cases}$$

C. Bourdarias, M. Ersoy and S. Gerbi

A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.

Int. J. On Finite Volumes, 6(2) :1–47, 2009.

Mathematical properties

- The **PFS** system is strictly hyperbolic for $A(t,x) > 0$.
- For regular solutions, the mean speed $u = Q/A$ verifies

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + c^2 \ln(A/S) + g\,\mathcal{H}(S)\cos\theta + g\,Z \right) = -g\,K(x,\mathbf{S})\,u\,|u|$$

and for $u = 0$, we have :

$$c^2 \ln(A/\mathbf{S}) + g\,\mathcal{H}(\mathbf{S})\cos\theta + g\,Z = cte$$

where $\mathcal{H}(\mathbf{S})$ is the physical water height.

- There exists a mathematical entropy

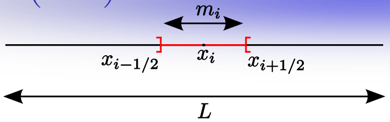
$$E(A,Q,S) = \frac{Q^2}{2A} + c^2 A \ln(A/\mathbf{S}) + c^2 S + g\overline{z}(x,\mathbf{S})\cos\theta + gAZ$$

which satisfies

$$\partial_t E + \partial_x \left(E\,u + p(x,A,E)\,u\right) = -g\,A\,K(x,\mathbf{S})\,u^2\,|u| \leqslant 0$$

Outline

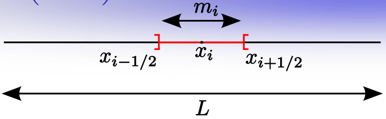
Finite Volume (VF) numerical scheme of order 1



PFS equations under vectorial form :

$$\partial_t \mathbf{U}(t,x) + \partial_x F(x,\mathbf{U}) = \mathcal{S}(t,x)$$

Finite Volume (VF) numerical scheme of order 1



PFS equations under vectorial form :

$$\partial_t \mathbf{U}(t,x) + \partial_x F(x,\mathbf{U}) = \mathcal{S}(t,x)$$

with $\mathbf{U}_i^n \overset{\text{cte per mesh}}{\approx} \dfrac{1}{\Delta x}\displaystyle\int_{m_i} \mathbf{U}(t_n,x)\,dx$ and $\mathcal{S}(t,x)$ constant per mesh,

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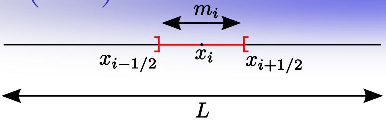

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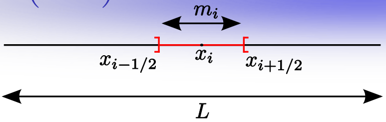
Cell-centered numerical scheme :

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right) + \Delta t^n \mathcal{S}(\mathbf{U}_i^n)$$

where

$$\Delta t^n \mathcal{S}_i^n \approx \int_{t_n}^{t_{n+1}}\int_{m_i} \mathcal{S}(t,x)\,dx\,dt$$

FINITE VOLUME (VF) NUMERICAL SCHEME OF ORDER 1



PFS equations under vectorial form :

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with $\mathbf{U}_i^n \overset{\text{cte per mesh}}{\approx} \dfrac{1}{\Delta x}\displaystyle\int_{m_i} \mathbf{U}(t_n,x)\,dx$ and $\mathcal{S}(t,x)$ constant per mesh,

Upwinded numerical scheme :

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x}\left(\widetilde{\mathcal{F}}_{i+1/2} - \widetilde{\mathcal{F}}_{i-1/2}\right)$$

$\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are consistent.

Choice of the numerical fluxes

Our goal : define $\mathcal{F}_{i+1/2}$ in order to preserve continuous properties of the PFS-model

Positivity of A ,

conservativity of A, discrete equilibrium, discrete entropy inequality

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Our choice

VFRoe solver[BEGVF]

Kinetic solver[BEG10]

C. Bourdarias, M. Ersoy and S. Gerbi.
A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.
International Journal On Finite Volumes , Vol 6(2) 1–47, 2009.

C. Bourdarias, M. Ersoy and S. Gerbi.
A kinetic scheme for transient mixed flows in non uniform closed pipes : a global manner to upwind all the source terms.
J. Sci. Comp.,pp 1-16, 10.1007/s10915-010-9456-0, 2011.

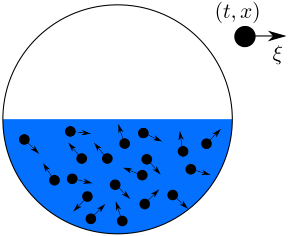
OUTLINE

Philosophy

As in kinetic theory of gases,

Describe the macroscopic behavior from particle motions, here, assumed fictitious

by introducing $\begin{cases} \text{a } \chi \text{ density function and} \\ \text{a } \mathcal{M}(t,x,\xi;\chi) \text{ maxwellian function (or a Gibbs equilibrium)} \end{cases}$



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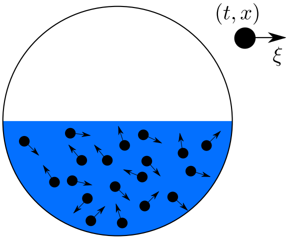

i.e., transform the nonlinear system into a kinetic transport equation on $\mathcal{M}$.

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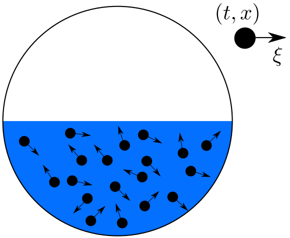

i.e., transform the nonlinear system into a kinetic transport equation on $\mathcal{M}$.
Thus, to be able to define the numerical *macroscopic fluxes* from the microscopic one.

...Faire d'une pierre deux coups...

We introduce

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1 \ ,$$

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then we define the Gibbs equilibrium by

$$\mathcal{M}(t,x,\xi) = \frac{A(t,x)}{b(t,x)}\, \chi\left(\frac{\xi - u(t,x)}{b(t,x)}\right)$$

with

$$b(t,x) = \sqrt{\frac{p(t,x)}{A(t,x)}}$$

Principle

Since

$$\chi(\omega) = \chi(-\omega) \geq 0 \;, \; \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1 \;,$$

and

$$\mathcal{M}(t,x,\xi) = \frac{A(t,x)}{b(t,x)} \chi\left(\frac{\xi - u(t,x)}{b(t,x)}\right)$$

then

MICRO-MACROSCOPIC RELATIONS

$$\begin{aligned} A &= \int_{\mathbb{R}} \mathcal{M}(t,x,\xi)\, d\xi \\ Q &= \int_{\mathbb{R}} \xi \mathcal{M}(t,x,\xi)\, d\xi \\ \frac{Q^2}{A} + \underbrace{A\, b^2}_{p} &= \int_{\mathbb{R}} \xi^2 \mathcal{M}(t,x,\xi)\, d\xi \end{aligned}$$

Principle [P02]

The kinetic formulation

(A,Q) is solution of the PFS system if and only if $\mathcal{M}$ satisfy the transport equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\Phi\, \partial_\xi \mathcal{M} = \mathcal{K}(t,x,\xi)$$

where $\mathcal{K}(t,x,\xi)$ is a collision kernel satisfying a.e. (t,x)

$$\int_{\mathbb{R}} \mathcal{K}\, d\xi = 0\ ,\ \int_{\mathbb{R}} \xi\, \mathcal{K} d\,\xi = 0$$

and Φ are the source terms.

B. Perthame.
Kinetic formulation of conservation laws.
Oxford University Press.
Oxford Lecture Series in Mathematics and its Applications, Vol 21, 2002.

Principe

The kinetic formulation

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and Φ are the source terms.

General form of the source terms :

$$\Phi = \overbrace{\frac{d}{dx} Z}^{\text{conservative}} + \overbrace{\mathbf{B} \cdot \frac{d}{dx} \mathbf{W}}^{\text{non conservative}} + \overbrace{K \frac{Q|Q|}{A^2}}^{\text{friction}}$$

with $\mathbf{W} = (Z, S, \cos\theta)$

Discretization of source terms

- Recalling that A, Q and $Z, S, \cos\theta$ constant per mesh
- forgetting the friction : « splitting »...

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Then $\forall (t,x) \in [t_n, t_{n+1}[\times \overset{\circ}{m}_i$

$$\Phi(t,x) = 0$$

since

$$\Phi = \frac{d}{dx} Z + \mathbf{B} \cdot \frac{d}{dx} \mathbf{W}$$

Simplification of the transport equation

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$\Longrightarrow$

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} = \mathcal{K}(t,x,\xi)$$

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$\Longrightarrow$

$$\left\{ \begin{array}{lll} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}(t_n, x, \xi) \overset{def}{:=} \dfrac{A(t_n,x,\xi)}{b(t_n,x,\xi)} \chi \left(\dfrac{\xi - u(t_n,x,\xi)}{b(t_n,x,\xi)} \right) \end{array} \right.$$

by neglecting the collision kernel.

Discretization of source terms

On $[t_n, t_{n+1}[\times m_i$, we have :

$$\left\{ \begin{array}{lll} \partial_t f + \xi \cdot \partial_x f & = & 0 \\ f(t_n, x, \xi) & = & \mathcal{M}_i^n(\xi) \end{array} \right.$$

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i.e.

$$f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \xi \frac{\Delta t^n}{\Delta x} \left(\mathcal{M}_{i+\frac{1}{2}}^-(\xi) - \mathcal{M}_{i-\frac{1}{2}}^+(\xi) \right)$$

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where

$$\mathbf{U}_i^{n+1} = \left(\begin{array}{c} A_i^{n+1} \\ Q_i^{n+1} \end{array} \right) \overset{def}{:=} \int_{\mathbb{R}} \left(\begin{array}{c} 1 \\ \xi \end{array} \right) f_i^{n+1}(\xi) \, d\xi$$

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or

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t^n}{\Delta x} \left(\widetilde{\mathcal{F}}_{i+1/2}^- - \widetilde{\mathcal{F}}_{i-1/2}^+ \right)$$

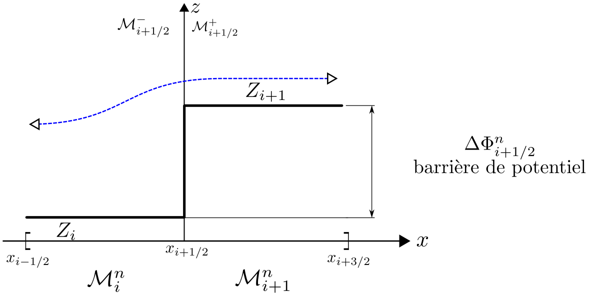
with

$$\widetilde{\mathcal{F}}_{i\pm\frac{1}{2}}^{\pm} = \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}_{i\pm\frac{1}{2}}^{\pm}(\xi)\, d\xi.$$

The microscopic fluxes

Interpretation : potential bareer

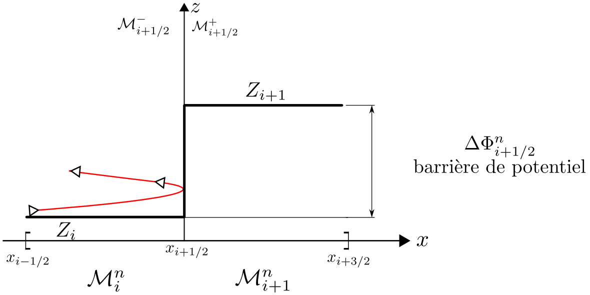
$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}^n_i(\xi)}^{\text{positive transmission}}$$
$$+ \underbrace{\mathbb{1}_{\{\xi<0,\, \xi^2-2g\Delta\Phi^n_{i+1/2}>0\}}\mathcal{M}^n_{i+1}\left(-\sqrt{\xi^2 - 2g\Delta\Phi^n_{i+1/2}}\right)}_{\text{negative transmission}}$$



The microscopic fluxes

Interpretation : potential bareer

$$\mathcal{M}^-_{i+1/2}(\xi) = \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}^n_i(\xi)}^{\text{positive transmission}} + \overbrace{\mathbb{1}_{\{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}<0\}}\mathcal{M}^n_i(-\xi)}^{\text{reflection}}$$
$$+ \underbrace{\mathbb{1}_{\{\xi<0,\,\xi^2-2g\Delta\Phi^n_{i+1/2}>0\}}\mathcal{M}^n_{i+1}\left(-\sqrt{\xi^2-2g\Delta\Phi^n_{i+1/2}}\right)}_{\text{negative transmission}}$$



THE MICROSCOPIC FLUXES

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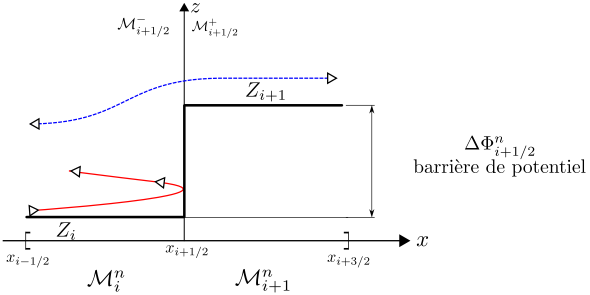

$\Delta\Phi^n_{i+1/2}$ may be interpreted as a time-dependent slope !

The microscopic fluxes

Interpretation : Dynamic slope $\Longrightarrow$ Upwinding of the friction

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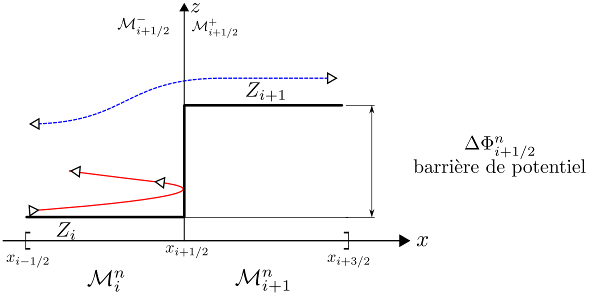

$\Delta\Phi^n_{i+1/2}$ may be interpreted as a time-dependent slope !

. . . we reintegrate the friction . . .

UPWINDING OF THE SOURCE TERMS : $\Delta\Phi_{i+1/2}$

- conservative $\partial_x \mathbf{W}$:

$$\mathbf{W}_{i+1} - \mathbf{W}_i$$

- non-conservative $\mathbf{B}\partial_x \mathbf{W}$:

$$\overline{\mathbf{B}}(\mathbf{W}_{i+1} - \mathbf{W}_i)$$

where

$$\overline{\mathbf{B}} = \int_0^1 \mathbf{B}(s, \phi(s, \mathbf{W}_i, \mathbf{W}_{i+1}))\, ds$$

for the « straight lines paths », i.e.

$$\phi(s, \mathbf{W}_i, \mathbf{W}_{i+1}) = s\mathbf{W}_{i+1} + (1-s)\mathbf{W}_i,\ s \in [0,1]$$

G. Dal Maso, P. G. Lefloch and F. Murat.
Definition and weak stability of nonconservative products.
J. Math. Pures Appl. , Vol 74(6) 483–548, 1995.

OUTLINE

χ =??? IN PRACTICE ? ? ?

Let us recall that we have to define a χ function such that :

$$\chi(\omega) = \chi(-\omega) \geq 0\ ,\ \int_{\mathbb{R}} \chi(\omega)d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega)d\omega = 1\ ,$$

and $\mathcal{M} = \frac{A}{b}\chi\left(\frac{\xi - u}{b}\right)$ satisfies the equation :

$$\partial_t \mathcal{M} + \xi \cdot \partial_x \mathcal{M} - g\Phi\, \partial_\xi \mathcal{M} = 0$$

and

$\chi \longrightarrow$ definition of the macroscopic fluxes.

PROPERTIES RELATED TO χ

We always have

- Conservativity of A holds for every χ.
- Positivity of A holds for every χ but for numerical purpose iff $\mathrm{supp}\chi$ is compact to get a CFL condition.

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strongly depend on the choice of the χ function.

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strongly depend on the choice of the χ function.

In the following, we only focus on discrete equilibrium.

OUTLINE

1. UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)
   - Previous works
   - Formal derivation of the free surface and pressurized model
   - A coupling : the PFS-model
2. A FINITE VOLUME FRAMEWORK
   - Kinetic Formulation and numerical scheme
   - The χ function and well balanced scheme
     1. Classical scheme fails in presence of complex source terms
     2. An alternative toward a Well-Balanced scheme
   - Numerical results
3. CONCLUSION AND PERSPECTIVES

STRATEGY

Even if the pipe is circular with uniform cross-sections, for instance for the free surface flows, the following procedure fails for complex source terms :

Following [PS01], choose χ such that $\mathcal{M}(t,x,\xi;\chi)$ is the steady state solution at rest, $u = 0$:

$$\xi \cdot \partial_x \mathcal{M} - g\Phi\, \partial_\xi \mathcal{M} = 0.$$

provides

$$\frac{3\,T\,I_1 - A^2}{2\,I_1} w\chi(w) + \left\{ \frac{A^2}{I_1} - w^2 \frac{A^2 - I_1\,T}{2\,I_1} \right\} \chi'(w) = 0 \text{ where } w = \frac{\xi}{b}\,.$$

B. Perthame and C. Simeoni
A kinetic scheme for the Saint-Venant system with a source term.
Calcolo, 38(4) :201–231, 2001.

Strategy

Even if the pipe is circular with uniform cross-sections, for instance for the free surface flows, the following procedure fails for complex source terms :
Following [PS01], choose χ such that $\mathcal{M}(t,x,\xi;\chi)$ is the steady state solution at rest, $u=0$:

$$\xi \cdot \partial_x \mathcal{M} - g\Phi\, \partial_\xi \mathcal{M} = 0.$$

provides

$$\underbrace{\frac{3\,T\,I_1 - A^2}{2\,I_1}}_{\alpha} w\chi(w) + \left\{ \underbrace{\frac{A^2}{I_1}}_{\beta} - w^2 \underbrace{\frac{A^2 - I_1\,T}{2\,I_1}}_{\gamma} \right\} \chi'(w) = 0\,.$$

Then, this equation is solvable as an ODE iff the coefficients (α,β,γ) are constants.

B. Perthame and C. Simeoni
A kinetic scheme for the Saint-Venant system with a source term.
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$$\xi \cdot \partial_x \mathcal{M} - g\Phi\, \partial_\xi \mathcal{M} = 0.$$

provides

$$\underbrace{\frac{3\,T\,I_1 - A^2}{2\,I_1}}_{\alpha} w\chi(w) + \left\{ \underbrace{\frac{A^2}{I_1}}_{\beta} - w^2 \underbrace{\frac{A^2 - I_1\,T}{2\,I_1}}_{\gamma} \right\} \chi'(w) = 0\,.$$

Then, this equation is solvable as an ODE iff the coefficients (α,β,γ) are constants.

For a rectangular pipe with uniform sections, we have $(\alpha,\beta,\gamma) = \left(\frac{T}{2}, 2T, \frac{T}{2}\right)$ with $T = cst$ the base of the pipe.

B. Perthame and C. Simeoni
A kinetic scheme for the Saint-Venant system with a source term.
Calcolo, 38(4) :201–231, 2001.

In these settings

With $(\alpha, \beta, \gamma) = \left(\frac{T}{2}, 2T, \frac{T}{2}\right)$ and

Theorem

we get $\chi(w) = \frac{1}{\pi}\left(1 - \frac{w^2}{4}\right)_+^{1/2}$ *and the numerical scheme satisfies the following properties :*

- *Positivity of A (under a CFL condition),*
- *Conservativity of A,*
- *Discrete equilibrium,*
- *Discrete entropy inequalities.*

- This results holds only for conservative terms $\partial_x Z(x)$.
- A similar result for pressurized flows, unusable in practice (see [PhDErsoy] Chap. 2).

M. Ersoy
Modeling, mathematical and numerical analysis of various compressible or incompressible flows in thin layer [Modélisation, analyse mathématique et numérique de divers écoulements compressibles ou incompressibles en couche mince].
Université de Savoie, Chambéry, September 10, 2010.

If (α, β, γ) are not constants . . .

Then, the equation to solve is :

$$\xi \cdot \partial_x \mathcal{M} - g\Phi \, \partial_\xi \mathcal{M} = 0.$$

Complicate to solve $\longrightarrow$ find an easy way to maintain, at least, discrete steady states.

Outline

1. Unsteady mixed flows : PFS equations (Pressurized and Free Surface)
   - Previous works
   - Formal derivation of the free surface and pressurized model
   - A coupling : the PFS-model

2. A Finite Volume Framework
   - Kinetic Formulation and numerical scheme
   - The χ function and well balanced scheme
     1. Classical scheme fails in presence of complex source terms
     2. An alternative toward a Well-Balanced scheme
   - Numerical results

3. Conclusion and perspectives

Correction of the macroscopic fluxes

The steady state is perfectly maintained iff

$$\widetilde{\mathcal{F}}^{-}_{i+1/2}(\mathbf{U}_i, \mathbf{U}_{i+1}, \mathbf{Z}_i, \mathbf{Z}_{i+1}) - \widetilde{\mathcal{F}}^{+}_{i-1/2}(\mathbf{U}_{i-1}, \mathbf{U}_i, \mathbf{Z}_{i-1}, \mathbf{Z}_i) = 0$$

with $\mathbf{U} = (A, Q)$, $\mathbf{Z} =$ "source terms"

Notations : $F_{i\pm1/2}$ the numerical flux of the homogeneous system, $\widetilde{F_{i\pm1/2}}$ the numerical flux with source term and F the flux of the PFS-model.

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with $\mathbf{U} = (A, Q)$, $\mathbf{Z} =$ "source terms"
Let us recall that without sources, whenever the numerical flux is consistent, i.e.

$$\forall \mathbf{U} = (A, Q) \in \mathbb{R}^2,\ F_{i\pm1/2}(\mathbf{U}, \mathbf{U}) = F(\mathbf{U}),$$

we automatically have, whenever steady states occurs :

$$F^{-}_{i+1/2}(\mathbf{U}_i, \mathbf{U}_{i+1}) - F^{+}_{i-1/2}(\mathbf{U}_{i-1}, \mathbf{U}_i) = 0,$$

i.e.,

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n.$$

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Correction of the macroscopic fluxes

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Correction of the numerical flux $\rightarrow$ toward a well balanced scheme

Notations : $F_{i\pm 1/2}$ the numerical flux of the homogeneous system, $\widetilde{F_{i\pm 1/2}}$ the numerical flux with source term and F the flux of the PFS-model.

IDEAS : replace

- dynamic quantities $\mathbf{U}_{i-1}$ and $\mathbf{U}_{i+1}$ by stationary profiles $\mathbf{U}^{+}_{i-1}$ and $\mathbf{U}^{-}_{i+1}$
- sources terms $\mathbf{Z}_{i-1}$ and $\mathbf{Z}_{i+1}$ by stationary profiles $\mathbf{Z}^{+}_{i-1}$ and $\mathbf{Z}^{-}_{i+1}$

Definition of the new fluxes : M-scheme

IDEAS : replace

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- sources terms $\mathbf{Z}_{i-1}$ and $\mathbf{Z}_{i+1}$ by stationary profiles $\mathbf{Z}_{i-1}^{+}$ and $\mathbf{Z}_{i+1}^{-}$

With A_{i+1}^{-} and A_{i-1}^{+} computed from the steady states :

$$\forall i, \begin{cases} D(A_{i+1}^{-}, Q_{i+1}, \mathbf{Z}_i) & = & D(\mathbf{U}_{i+1}, \mathbf{Z}_{i+1}) \\ D(A_{i-1}^{+}, Q_{i-1}, \mathbf{Z}_i) & = & D(\mathbf{U}_{i-1}, \mathbf{Z}_{i-1}) \end{cases}$$

where $D(\mathbf{U}, \mathbf{Z}) = \dfrac{Q^2}{2A} + \begin{cases} g\mathcal{H}(A)\cos\theta + gZ & \text{if } E = 0, \\ c^2 \ln\left(\dfrac{A}{S}\right) + g\mathcal{H}(S)\cos\theta + gZ & \text{if } E = 1. \end{cases}$

And $(\mathbf{Z}_{i+1}^{-}, \mathbf{Z}_{i-1}^{+})$ are defined as follows :

$$\mathbf{Z}_{i+1}^{-} = \begin{cases} \mathbf{Z}_i & \text{if} & A_{i+1}^{-} = A_i \\ \mathbf{Z}_{i+1} & \text{if} & A_{i+1}^{-} \neq A_i \end{cases}$$

$$\mathbf{Z}_{i-1}^{+} = \begin{cases} \mathbf{Z}_i & \text{if} & A_{i-1}^{+} = A_i \\ \mathbf{Z}_{i-1} & \text{if} & A_{i-1}^{+} \neq A_i \end{cases}$$

DEFINITION OF THE NEW FLUXES : M-SCHEME

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Let us now consider

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t^n}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}}^{-}(\mathbf{U}_i^n,\ A_{i+1}^{-}\ ,Q_{i+1}^n,\mathbf{Z}_i,\ \mathbf{Z}_{i+1}^{-}\ ) - \mathbf{F}_{i-\frac{1}{2}}^{+}(\ A_{i-1}^{+}\ ,Q_{i-1}^n,\mathbf{U}_i^n,\ \mathbf{Z}_{i-1}^{+}\ ,\mathbf{Z}_i)\right)$$

Definition of the new fluxes : M-scheme

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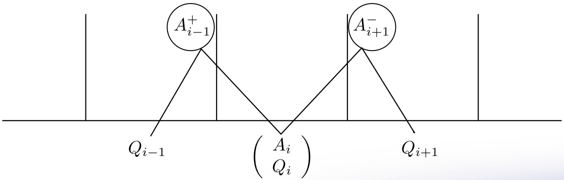
instead of the previous one :

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t^n}{\Delta x}\left(\mathbf{F}^{-}_{i+\frac{1}{2}}(\mathbf{U}_i^n, A^{n}_{i+1}, Q^n_{i+1}, \mathbf{Z}_i, \mathbf{Z}_{i+1}) - \mathbf{F}^{+}_{i-\frac{1}{2}}(A^{n}_{i-1}, Q^n_{i-1}, \mathbf{U}_i^n, \mathbf{Z}_{i-1}, \mathbf{Z}_i)\right)$$



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Let us now consider

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Then,

Theorem

the numerical scheme is well-balanced.

Proof

- the numerical flux is, by construction, consistent.

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- Let us assume that there exits n such that for every i :

$$Q_i^n = Q_0,\ D(\mathbf{U}_i^n, \mathbf{Z}_i) = h_0.$$

Proof

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Then,

$$D(A_{i+1}^-, Q_{i+1}, \mathbf{Z}_i) = D(\mathbf{U}_{i+1}, \mathbf{Z}_{i+1}) = h_0,\ \forall i$$

and especially, we have :

$$D(A_{i+1}^-, Q_{i+1}, \mathbf{Z}_i) = D(\mathbf{U}_i, \mathbf{Z}_i).$$

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$$D(A_{i+1}^-, Q_{i+1}, \mathbf{Z}_i) = D(\mathbf{U}_i, \mathbf{Z}_i).$$

The application $A \to D(A, Q, Z)$ being injective, provides $A_{i+1}^- = A_i$ and thus $\mathbf{Z}_{i+1}^- = \mathbf{Z}_i$ by construction. Similarly, we get $A_{i-1}^+ = A_i$ and $\mathbf{Z}_{i-1}^+ = \mathbf{Z}_i$.

Proof

- the numerical flux is, by construction, consistent.
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Then,

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The application $A \to D(A, Q, Z)$ being injective, provides $A_{i+1}^- = A_i$ and thus $\mathbf{Z}_{i+1}^- = \mathbf{Z}_i$ by construction. Similarly, we get $A_{i-1}^+ = A_i$ and $\mathbf{Z}_{i-1}^+ = \mathbf{Z}_i$.
Finally, since

$$\mathbf{F}_{i+\frac{1}{2}}^-(\mathbf{U}_i^n, \mathbf{U}_{i+1}^-, \mathbf{Z}_i, \mathbf{Z}_{i+1}^-) - \mathbf{F}_{i-\frac{1}{2}}^+(\mathbf{U}_{i-1}^+, \mathbf{U}_i^n, \mathbf{Z}_{i-1}^+, \mathbf{Z}_i) = 0,$$

we get $\forall l \geqslant n,\ \ Q_i^{l+1} = Q_i^l := Q_0.$

■

Numerical properties

For instance, with the simplest χ function [ABP00],

$$\chi(\omega) = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(\omega)$$

the following properties holds :

- Positivity of A (under a CFL condition),
- Conservativity of A,
- Discrete equilibrium and,
- Natural treatment of drying and flooding area.

for example

and analytical expression of the numerical macroscopic fluxes.

E. Audusse and M-0. Bristeau and B. Perthame.
Kinetic schemes for Saint-Venant equations with source terms on unstructured grids.
INRIA Report RR3989, 2000.

OUTLINE

1. UNSTEADY MIXED FLOWS : PFS EQUATIONS (PRESSURIZED AND FREE SURFACE)
   - Previous works
   - Formal derivation of the free surface and pressurized model
   - A coupling : the PFS-model
2. A FINITE VOLUME FRAMEWORK
   - Kinetic Formulation and numerical scheme
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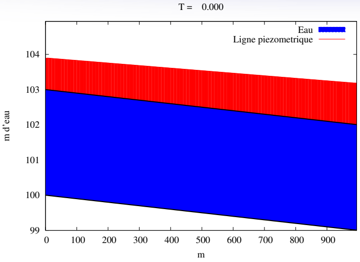
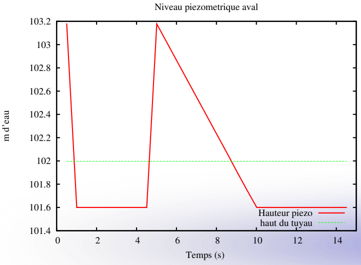
3. CONCLUSION AND PERSPECTIVES

Qualitative analysis of convergence

and comparison with the Well-Balanced VFRoe scheme



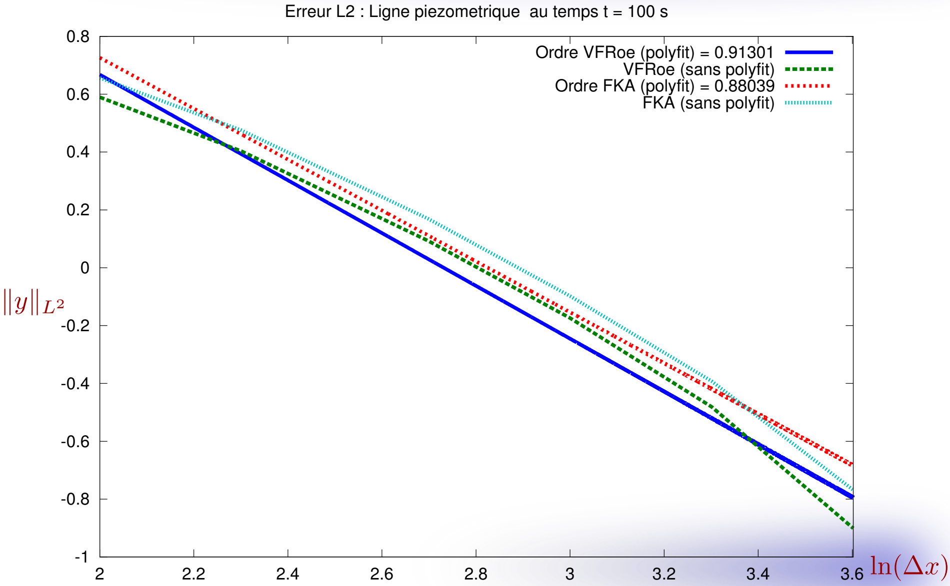
- upstream piezometric head $104\ m$



- downstream piezometric head :

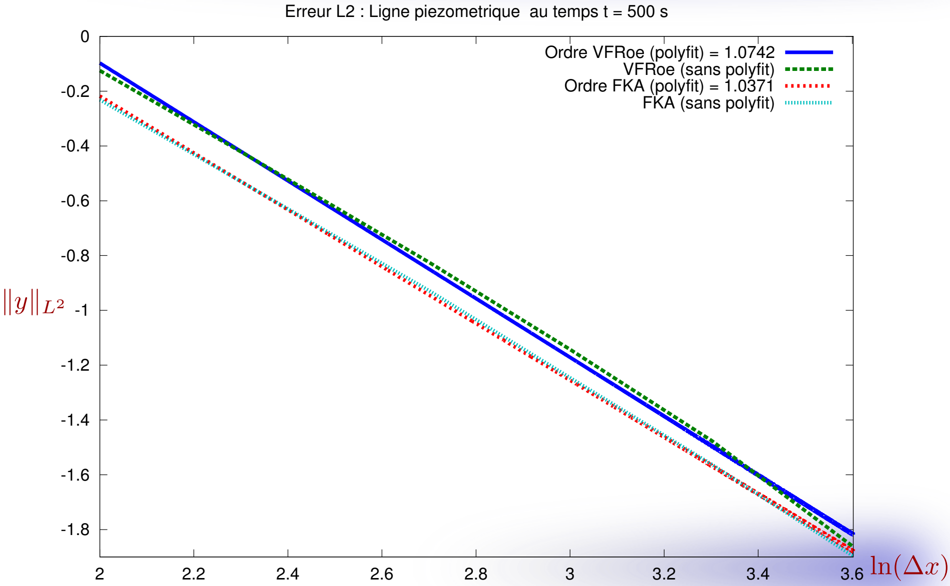
Convergence

During unsteady flows $t = 100\ s$



Convergence

Stationary $t = 500\ s$



Conclusion

✅ Conservative and simple formulation :

$\longrightarrow$ easy implementation even if source terms are complex

✅ The most of the properties of the continuous model are maintained at discrete level :

$\longrightarrow$ positivity of the water area

$\longrightarrow$ conservativity of the water area

$\longrightarrow$ discrete equilibrium maintained

Conclusion and perspectives

✅ Conservative and simple formulation :

⟶ easy implementation even if source terms are complex

✅ The most of the properties of the continuous model are maintained at discrete level :

⟶ positivity of the water area

⟶ conservativity of the water area

⟶ discrete equilibrium maintained

What about discrete entropy inequalities ?

⟶ same difficulties as for discrete balance (see [PhDErsoy] Chap. 2 for further details)

Thank you

for your

attention

2+3 tane dis ! ! ! 20 Mars 2011