# Compressible primitive equations

**M. Ersoy**, IMATH, Toulon [1]

*8èmes journées scientifiques de l'Université de Toulon*

Toulon,
April 16, 2014

1. joint work with T. Ngom and M. Sy (LANI, Senegal)

Outline of the talk

# Outline

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## Context : compressible equation in a thin layer domain

- Navier-Stokes equations on $\Omega = \{(\mathbf{x}, y) \in \mathbb{R}^2 \times \mathbb{R}; H \ll L\}$

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- Hydrostatic approximation $(\varepsilon = H/L = W/V \ll 1)$ $\longrightarrow$ Primitive equations

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# Context : compressible equation in a thin layer domain

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- Hydrostatic approximation $(\varepsilon = H/L = W/V \ll 1) \longrightarrow$ Primitive equations

$\downarrow$ [EP14]

- Averaged Primitive Equations wr to $y \longrightarrow$ Pressurized model


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*M. Ersoy*
A pressurized model for compressible pipe flow.
*Submitted, 2014.*

# Atmosphere dynamic : assumptions

- Dynamic :

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  - Compressible fluid

- Modeling (neglecting phenomena such as the evaporation and solar heating) : Compressible Navier-Stokes equations

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\rho + \rho \mathrm{div}\mathbf{U} & = & 0 \\ \rho\dfrac{d}{dt}\mathbf{u} + \nabla_x p & = & \mathrm{div}_x(\sigma_x) + f \\ \partial_t(\rho v) + \mathrm{div}(\rho \mathbf{U} v) + \partial_y p(\rho) & = & -\rho g + \mathrm{div}_y(\sigma_y) \\ p(\rho) & = & c^2\,\rho \end{array}\right.$$

with $\mathbf{U} = (\mathbf{u}, v)$, $\dfrac{d}{dt} := \partial_t + \mathbf{u}\cdot\nabla_x + v\partial_y$, $\sigma := 2\Sigma.D(\mathbf{U}) + \lambda \mathrm{div}(\mathbf{U})$

- Dynamic :
  - Compressible fluid
  - vertical scale $\ll$ horizontal scale

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  Hydrostatic approximation $\longrightarrow$ Compressible Primitive Equations (CPEs)

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# Atmosphere dynamic : assumptions

- Dynamic :
  - Compressible fluid
  - vertical scale $\ll$ horizontal scale
  - Stratified density

$$p = \xi(t,x) e^{-g/c^2 y}$$

- Modeling (neglecting phenomena such as the evaporation and solar heating) : Compressible Navier-Stokes equations

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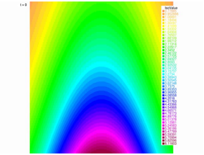
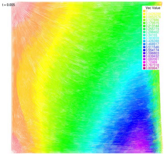
*M. Ersoy and T. Ngom*
Existence of a global weak solution to a Compressible Primitive Equations.
*C. R. Acad. Sci. Paris, Ser. I, http ://dx.doi.org/10.1016/j.crma.2012.04.013*, 2012.

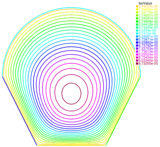
*M. Ersoy, T. Ngom and M. Sy*
Compressible primitive equations : formal derivation and stability of weak solutions.
*Nonlinearity*, 24(1), pp 79-96, 2011.

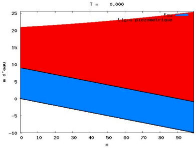
- Atmosphere modeling (left : density, right : velocity field)




- Pipe flow modeling (left : velocity profile in a "horse shoe" pipe, right : piezometric line)



Outline

## THE TWO-DIMENSIONAL PROBLEM [EN10]

Let us consider the equations on $\Omega = \{(x,y);\, 0 < x < l,\, 0 < y < h\}$

$$\left\{\begin{array}{rcl} \partial_t\rho + \partial_x(\rho\mathbf{u}) + \partial_y(\rho v) & = & 0 \\ \partial_t(\rho\mathbf{u}) + \partial_x(\rho\mathbf{u}^2) + \partial_y(\rho\mathbf{u}v) + c^2\partial_x\rho & = & \partial_x(\nu_1(t,x,y)\partial_x\mathbf{u}) \\ & & +\partial_y(\nu_2(t,x,y)\partial_y\mathbf{u}) \\ c^2\partial_y\rho & = & -\rho g \end{array}\right.$$

with the following boundary conditions :

$$\mathbf{u}_{|x=0} = \mathbf{u}_{|x=l} = 0, \quad v_{|y=0} = v_{|y=h} = 0, \quad \partial_y\mathbf{u}_{|y=0} = \partial_y\mathbf{u}_{|y=h} = 0$$

and the initial data

$$u_{|t=0}(x,y) = u_0(x,y), \quad \rho_{|t=0}(x,y) = \xi_0(x)e^{-g/c^2 y}, \text{with} \quad 0 < m \leqslant \xi_0 \leqslant M < \infty$$

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keeping in mind the stratified structure of $\rho = \xi(t,x)e^{-g/c^2 y}$ and


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keeping in mind the stratified structure of $\rho = \xi(t,x) e^{-g/c^2 y}$ and assume

$$\nu_1(t,x,y) = \overline{\nu_1} e^{-\frac{g}{c^2} y}, \quad \nu_2(t,x,y) = \overline{\nu_2} e^{\frac{g}{c^2} y}, \text{ with } (\overline{\nu_1}, \overline{\nu_2}) \in \mathbb{R}_+^2 \ .$$

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## A useful change of variables [EN10]

Multiplying the previous equations by $e^{g/c^2 y}$, we get :

$$
\left\{
\begin{array}{rcl}
\partial_t\left(\xi\right)+\partial_x\left(\xi\mathbf{u}\right)+e^{g/c^2y}\partial_y\left(\xi e^{-g/c^2y}v\right) & = & 0\\
\partial_t\left(\xi\mathbf{u}\right)+\partial_x\left(\xi\mathbf{u}^2\right)+e^{g/c^2y}\partial_y\left(\xi\mathbf{u}e^{-g/c^2y}v\right)+c^2\partial_x\left(\xi\right) & = & \partial_x(\overline{\nu_1}\partial_x\mathbf{u})+\\
 & & e^{g/c^2y}\partial_y\left(\overline{\nu_2}e^{g/c^2y}\partial\right.\\
c^2\partial_y\left(\xi e^{-g/c^2y}\right) & = & -\xi e^{-g/c^2y}g
\end{array}
\right.
$$

setting

1. $w = e^{-g/c^2 y} v$


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# A useful change of variables [EN10]

Multiplying the previous equations by $e^{g/c^2 y}$, we get :

$$\begin{cases} \partial_t(\xi) + \partial_x(\xi\mathbf{u}) + e^{g/c^2 y}\partial_y(\xi w) &= 0 \\ \partial_t(\xi\mathbf{u}) + \partial_x\left(\xi\mathbf{u}^2\right) + e^{g/c^2 y}\partial_y(\xi\mathbf{u}w) + c^2\partial_x(\xi) &= \partial_x(\overline{\nu_1}\partial_x\mathbf{u}) + \\ & \quad e^{g/c^2 y}\partial_y\left(\overline{\nu_2}e^{g/c^2 y}\partial_y\mathbf{u}\right) \\ c^2\partial_y\left(\xi e^{-g/c^2 y}\right) &= -\xi e^{-g/c^2 y} g \end{cases}$$

setting

1. $w = e^{-g/c^2 y} v$
2. $e^{g/c^2 y}\partial_y = \partial_z$ with $z = 1 - \dfrac{c^2}{g} e^{-g/c^2 y}$


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Multiplying the previous equations by $e^{g/c^2 y}$, we get :

$$\left\{\begin{array}{rcl} \partial_t(\xi) + \partial_x(\xi \mathbf{u}) + \partial_z(\xi w) & = & 0 \\ \partial_t(\xi \mathbf{u}) + \partial_x\left(\xi \mathbf{u}^2\right) + \partial_z(\xi \mathbf{u} w) + c^2 \partial_x(\xi) & = & \partial_x(\overline{\nu_1}\partial_x \mathbf{u}) + \\ & & \partial_z(\overline{\nu_2}\partial_z \mathbf{u}) \\ \partial_z \xi & = & 0 \end{array}\right.$$

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# A useful change of variables [EN10]

Finally, we get a simplified compressible primitive model (SCP) on
$\Omega = \{(x,z);\, 0 < x < l,\, \bar{0} = 1 - c^2/g < y < H = 1 - c^2/ge^{-g/c^2 h}\}$

$$\left\{ \begin{array}{rcl} \partial_t\left(\xi\right) + \partial_x\left(\xi \mathbf{u}\right) + \partial_z\left(\xi w\right) & = & 0 \\ \partial_t\left(\xi \mathbf{u}\right) + \partial_x\left(\xi \mathbf{u}^2\right) + \partial_z\left(\xi \mathbf{u} w\right) + c^2 \partial_x\left(\xi\right) & = & \partial_x(\overline{\nu_1}\partial_x \mathbf{u}) + \\ & & \partial_z\left(\overline{\nu_2}\partial_z \mathbf{u}\right) \\ \partial_z \xi & = & 0 \end{array} \right.$$

with the following boundary conditions :

$$\mathbf{u}_{|x=0} = \mathbf{u}_{|x=l} = 0, \quad w_{|z=\bar{0}} = w_{|z=H} = 0, \quad \partial_z \mathbf{u}_{|z=\bar{0}} = \partial_z \mathbf{u}_{|z=H} = 0$$

and the initial data

$$u_{|t=0}(x,z) = u_0(x,z), \quad \xi_{|t=0}(x) = \xi_0(x), \text{with} \quad 0 < m \leqslant \xi_0 \leqslant M < \infty\ .$$

*N. E. Kochin*
On simplification of the equations of hydromechanics in the case of the general circulation of the atmosphere.
*Trudy, Glavn. Geofiz. Observator.*, 4 :21–45, 1936.

# As a consequence, firstly,

- Assume $c^2 = g = h = 1$,
- note $\overline{\mathbf{u}} = \dfrac{1}{H}\displaystyle\int_0^H \mathbf{u}(t,x,z)\,dz$,
- note $\xi$ for $\xi H$ and $\overline{\nu_1}$ for $\overline{\nu_1} H$

then the averaged (SCP) (ASCP) system becomes

$$\left\{\begin{array}{rcl} \partial_t(\xi) + \partial_x(\xi\overline{\mathbf{u}}) & = & 0 \\ \partial_t(\xi\overline{\mathbf{u}}) + \partial_x(\xi\overline{(\mathbf{u}^2)}) + \partial_x(c^2\xi) & = & \partial_x(\overline{\nu_1}\partial_x\overline{\mathbf{u}}) \end{array}\right.$$

using the boundary conditions :

$$\mathbf{u}_{|x=0} = \mathbf{u}_{|x=l} = 0, \quad w_{|z=\bar{0}} = w_{|z=H} = 0, \quad \partial_z\mathbf{u}_{|z=\bar{0}} = \partial_z\mathbf{u}_{|z=H} = 0$$

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- mean-oscillation $\mathbf{u} = \overline{\mathbf{u}} + \widetilde{\mathbf{u}}$ with $\displaystyle\int_0^H \widetilde{\mathbf{u}}dz = 0$

then the averaged (SCP) (ASCP) system becomes

$$\left\{\begin{array}{rcl} \partial_t(\xi) + \partial_x(\xi\overline{\mathbf{u}}) & = & 0 \\ \partial_t(\xi\overline{\mathbf{u}}) + \partial_x(\xi\overline{(\mathbf{u}^2)}) + \partial_x(c^2\xi) & = & \partial_x(\overline{\nu_1}\partial_x\overline{\mathbf{u}}) \end{array}\right.$$

using the boundary conditions :

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then the averaged (SCP) (ASCP) system becomes

$$\left\{\begin{array}{rcl} \partial_t(\xi) + \partial_x(\xi\overline{\mathbf{u}}) & = & 0 \\ \partial_t(\xi\overline{\mathbf{u}}) + \partial_x(\xi\overline{\mathbf{u}}^2) + \partial_x(c^2\xi + \xi\varphi(\widetilde{\mathbf{u}})) & = & \partial_x(\overline{\nu_1}\partial_x\overline{\mathbf{u}}) \end{array}\right.$$

with $\varphi(f) = \displaystyle\int_0^H f^2\,dz$.


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# As a consequence, secondly,

- multiplying the SCP model by $H$ (using the previous notations)

the SCP system becomes

$$\left\{\begin{array}{rcl} \partial_t(\xi) + \partial_x(\xi \mathbf{u}) + \partial_z(\xi w) & = & 0 \\ \partial_t(\xi \mathbf{u}) + \partial_x(\xi \mathbf{u}^2) + \partial_z(\xi \mathbf{u} w) + \partial_x(c^2 \xi) & = & D \\ \partial_z(\xi) & = & 0 \end{array}\right.$$

with $D = \partial_x(\overline{\nu_1}\partial_x \mathbf{u}) + \partial_z(\overline{\nu_2}\partial_z \mathbf{u})$


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- inserting $\mathbf{u} = \overline{\mathbf{u}} + \widetilde{\mathbf{u}}$ in the SCP model

the SCP system becomes

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$$\partial_t(\xi\overline{\mathbf{u}}) + \partial_x(\xi\overline{\mathbf{u}}^2) + \partial_x(c^2\xi) - \partial_x(\overline{\nu_1}\partial_x\overline{\mathbf{u}}) = -\partial_x(\xi\varphi(\widetilde{\mathbf{u}}))$$


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with $D = \partial_x(\overline{\nu_1}\partial_x\widetilde{\mathbf{u}}) + \partial_z(\overline{\nu_2}\partial_z\widetilde{\mathbf{u}})$

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- keeping in mind the ASCP model

the SCP system becomes

$$\left\{\begin{array}{rcl} \partial_x(\xi\widetilde{\mathbf{u}}) + \partial_z(\xi w) & = & 0 \\ \partial_t(\xi\widetilde{\mathbf{u}}) + \partial_x(\xi(2\overline{\mathbf{u}}\widetilde{\mathbf{u}} + \widetilde{\mathbf{u}}^2)) - \partial_x(\xi\varphi(\widetilde{\mathbf{u}})) + \partial_z(\xi\mathbf{u}w) & = & D \\ \partial_z(\xi) & = & 0 \end{array}\right.$$

with $D = \partial_x(\overline{\nu_1}\partial_x\widetilde{\mathbf{u}}) + \partial_z(\overline{\nu_2}\partial_z\widetilde{\mathbf{u}})$ i.e.

$$\partial_t(\xi\widetilde{\mathbf{u}}) + \partial_x(\xi(2\overline{\mathbf{u}}\widetilde{\mathbf{u}} + \widetilde{\mathbf{u}}^2) - \xi\varphi(\widetilde{\mathbf{u}})) + \partial_z(\xi\mathbf{u}w) \quad = \quad D$$

with $w = -\frac{1}{\xi}\int_0^H \partial_x(\xi\widetilde{\mathbf{u}})\,dz$ .

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# Finally,

Formally, solving the SCP model is then equivalent to find a fix point to :

- Given $\widetilde{\mathbf{u}}_1$, find $(\xi, \overline{\mathbf{u}}) := T_1(\widetilde{\mathbf{u}}_1)$ such that

$$\left\{\begin{array}{rcl} \partial_t(\xi) + \partial_x(\xi\overline{\mathbf{u}}) & = & 0 \\ \partial_t(\xi\overline{\mathbf{u}}) + \partial_x(\xi\overline{\mathbf{u}}^2) + \partial_x(c^2\xi + \xi\varphi(\widetilde{\mathbf{u}})) & = & \partial_x(\overline{\nu_1}\partial_x\overline{\mathbf{u}}) \end{array}\right.$$

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# Finally,

Formally, solving the SCP model is then equivalent to find a fix point to :

- Given $\widetilde{\mathbf{u}}_1$, find $(\xi, \overline{\mathbf{u}}) := T_1(\widetilde{\mathbf{u}}_1)$ such that

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- Given $(\xi, \overline{\mathbf{u}})$, find $\widetilde{\mathbf{u}}_2 = T_2(\xi, \overline{\mathbf{u}})$ such that

$$\partial_t(\xi\widetilde{\mathbf{u}}) + \partial_x(\xi(2\overline{\mathbf{u}}\widetilde{\mathbf{u}} + \widetilde{\mathbf{u}}^2) - \xi\varphi(\widetilde{\mathbf{u}})) + \partial_z(\xi\mathbf{u}w) \quad = \quad \partial_x(\overline{\nu_1}\partial_x\widetilde{\mathbf{u}}) + \partial_z(\overline{\nu_2}\partial_z\widetilde{\mathbf{u}})$$


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# Schauder fix point

Following [GK05] (suitable *a priori* estimates for $u$ (hence for $\widetilde{\mathbf{u}}$) and $\xi$ bounded), one can define the space

$$K = \left\{ f;\ \|f\|_{L^2(0,T,H^1(\Omega)} \leqslant k < \infty \right\}$$


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- providing $\widetilde{\mathbf{u}}_1 \in K$, one can construct $(\xi, \overline{\mathbf{u}})$ such that $T_1(\widetilde{\mathbf{u}}_1) = (\xi, \overline{\mathbf{u}})$ (see [AKM83]),


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- providing $(\xi, \overline{\mathbf{u}})$, one can construct $\widetilde{\mathbf{u}}_2 \in K$ such that $T_2(\xi, \overline{\mathbf{u}}) = \widetilde{\mathbf{u}}_2 \in K$.


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- providing $(\xi, \overline{\mathbf{u}})$, one can construct $\widetilde{\mathbf{u}}_2 \in K$ such that $T_2(\xi, \overline{\mathbf{u}}) = \widetilde{\mathbf{u}}_2 \in K$.

As a consequence, the operator $T = T_2 \circ T_1$ has at least one fixed point (Schauder fixed point theorem).


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## More precisely

Following [GK05],

### Definition

let us define $(\xi, \mathbf{u}, w)$ a weak solution of the SCP equations (i.e. satisfying SCPE in the distribution sense) such that

$$\xi \in L^\infty(0,T;H^1(0,H)), \quad \partial_t \xi \in L^2(0,T;L^2(0,H))$$

$$\mathbf{u} \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad w \in L^2(0,T;L^2(\Omega)) \ .$$


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Then,

### Theorem

*For every $(\xi_0, \mathbf{u}_0)$, there exists a weak solution to SCP where $\xi$ is a bounded strictly positive function.*


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## More precisely

Following [GK05],

### Definition

let us define $(\xi, \mathbf{u}, w)$ a weak solution of the SCP equations (i.e. satisfying CPE in the distribution sense) such that

$$\rho \in L^\infty(0,T;H^1(\Omega)), \quad \partial_t\rho \in L^2(0,T;L^2(\Omega))$$

$$\mathbf{u} \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad v \in L^2(0,T;L^2(\Omega))\ .$$

Then,

### Theorem

*For every $(\rho_0 = \xi_0 e^{-y}, \mathbf{u}_0)$, there exists a weak solution to SCP and therefore to the CP equations where $\rho$ is a bounded strictly positive function.*


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# Numerical approximation

- $t^{n+1} = t^n + \delta t$ be the discrete time with $\delta t$ be the time step,
- $U = (\mathbf{u}, w)^t$ and $\dfrac{d}{dt} = \partial_t + U \cdot \nabla$.

Let us write the SCP equations as follows :

$$\begin{aligned} \frac{d}{dt}\xi &= -\xi \mathrm{div}(U) && (1)\\ \xi \frac{d}{dt}\mathbf{u} &= -\partial_x(c^2\xi) + D && (2)\\ \partial_z(\xi) &= 0 && (3) \end{aligned}$$

with

$$D = \partial_x(\overline{\nu_1}\partial_x \mathbf{u}) + \partial_z(\overline{\nu_1}\partial_z \mathbf{u})$$

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Let us write the SCP equations as follows :

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$$\xi \frac{d}{dt}\mathbf{u} = -\partial_x(c^2\xi) + D \quad (2)$$

$$\partial_z(\xi) = 0 \quad (3)$$

with

$$D = \partial_x(\overline{\nu_1}\partial_x \mathbf{u}) + \partial_z(\overline{\nu_1}\partial_z \mathbf{u})$$

and remark that using (3) with (1), one has

$$\partial_{zz} w = -\frac{1}{\xi}\partial_x(\xi \partial_z \mathbf{u})$$

- $t^{n+1} = t^n + \delta t$ be the discrete time with $\delta t$ be the time step,
- $U = (\mathbf{u}, w)^t$ and $\dfrac{d}{dt} = \partial_t + U \cdot \nabla$.

Given $\xi^n$ and $\mathbf{u}^n$ at time $t^n$, one can approximate equations as follows :

$$
\begin{aligned}
\partial_{zz} w^n &= -\frac{1}{\xi^n}\partial_x(\xi^n \partial_z \mathbf{u}^n) \\
\frac{\xi^{n+1} - \xi^n \circ X^n}{\delta t} &= -\xi^n \mathrm{div}(U^n) \\
\frac{\mathbf{u}^{n+1} - \mathbf{u}^n \circ X^n}{\delta t} &= -\frac{1}{\xi^n}\left(\partial_x(c^2\xi^n) + D^n\right)
\end{aligned}
$$

with

$$
D = \partial_x(\overline{\nu_1}\partial_x \mathbf{u}) + \partial_z(\overline{\nu_1}\partial_z \mathbf{u}) \text{ and } \frac{d}{ds}X = U
$$

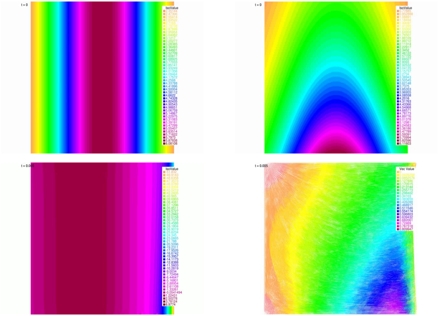
which can easily implemented in a FreeFem++ [a] code.

a. http://www.freefem.org/ff++/

# A NUMERICAL TEST

numerical parameters :

$$H = 1/2, g = c = 1, \delta t = 0.05, T_f = 2,5, \overline{\nu_1} = \overline{\nu_2} = 1, u_0(x,z) = xz$$



density $\xi$ (top left), density $\rho$ (top right), divergence $\mathrm{div}(\rho U)$ (bottom left), velocity field $U$ (bottom right),

# Outline

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1. uniqueness of the 2D problem ?

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2. existence and uniqueness of the 3D problem ?

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Partially ✅ [ENS11]


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1. uniqueness of the 2D problem ?
2. existence and uniqueness of the 3D problem ?

   Partially ☑ [ENS11]

3. what happens in the case $p(\rho) = \rho^\gamma, \gamma \neq 1$ ?


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# Thank you

# for your

# attention