On a new mathematical model for open channel and river hydraulics

Mehmet Ersoy

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Colloque Modèles asymptotiques et méthodes numériques pour les milieux continus et la biologie, Saint-Étienne, France

Outline of the talk

Outline

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- Introducing characteristic scales :
  - length L
  - width l
  - height H

- Introducing characteristic scales : L, l and H
- Introducing aspect ratio numbers :
  - $\varepsilon_z = \dfrac{H}{L}$ following the depth
  - $\varepsilon_y = \dfrac{l}{L}$ following the width

- Introducing characteristic scales : L, l and H
- Introducing aspect ratio numbers : $\varepsilon_z = \dfrac{H}{L}$ and $\varepsilon_y = \dfrac{l}{L}$
- One can reduce the initial model (Navier-Stokes or Euler equations)
  - 3D-2D depth averaged model reduction if

$$\varepsilon_z \ll 1 \text{ and } \varepsilon_y \approx 1$$

  - 3D-1D section averaged model reduction if

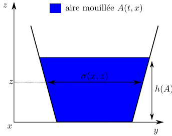
$$\varepsilon_z \approx \varepsilon_y \ll 1$$

- Introducing characteristic scales : L, l and H
- Introducing aspect ratio numbers :
- One can reduce the initial model (Navier-Stokes or Euler equations)
- Opposite to DNS, model reduction $\rightarrow$ to decrease the computational cost

Outline

SV equations

- 3D-1D model reduction for closed water pipes/channels/rivers



$$
\begin{cases}
\partial_t A + \partial_x Q = 0, \\
\partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + g I_1(x,A) \right) = g I_2(x,A)
\end{cases}
$$

with

$A(t,x)$, $Q(t,x)$, g, $h = \eta - d$	:	wet area, discharge, gravity
$I_1(x,A) = \int_d^\eta \sigma(x,z)(\eta - z)\,dz$	:	hydrostatic pressure
$I_2(x,A) = \int_d^\eta \dfrac{\partial}{\partial x} \sigma(x,z)(\eta - z)\,dz$	:	hydrostatic pressure source

C. Bourdarias, **M. Ersoy**, S. Gerbi.
A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.
International Journal on Finite Volumes, 2009.

C. Bourdarias, **M. Ersoy**, S. Gerbi.
A kinetic scheme for transient mixed flows in non uniform closed pipes : a global manner to upwind all the source terms.
Journal of Scientific Computing, 2011.

C. Bourdarias, **M. Ersoy**, S. Gerbi.
Unsteady mixed flows in non uniform closed water pipes : a Full Kinetic Appraoch.
Numerische Mathematik, 2014.

M. Ersoy.
Dimension reduction for incompressible pipe and open channel flow including friction.
Applications of Mathematics, 2015.

SV equations

- 3D-1D model reduction for closed water pipes/channels/rivers
- 3D-2D reduction for tsunamis propagation

$$\left\{\begin{array}{l}\partial_t h + \mathsf{div}(h\overline{u}) = 0,\\ \partial_t(h\overline{u}) + \mathsf{div}\left(h\overline{u}\otimes\overline{u} + g\dfrac{h^2}{2}I\right) = -gh\nabla Z,\end{array}\right.$$

with $\overline{u}(t,x)\in\mathbb{R}^2$: depth averaged velocity

K. Pons, **M. Ersoy**.
Adaptive mesh refinement method. Part 1 : Automatic thresholding based on a distribution function.
SEMA SIMAI Springer Series, Partial Differential Equations : Ambitious Mathematics for Real-Life Applications, D. Donatelli and C. Simeoni Editors, 2020

K. Pons, **M. Ersoy** , F. Golay and R. Marcer.
Adaptive mesh refinement method. Part 2 : Application to tsunamis propagation.
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- Tsunamis are water waves that start in the deep ocean : H is huge

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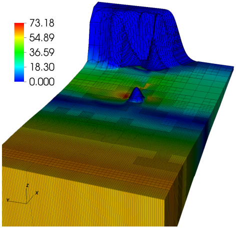
- But, the wavelength λ of the tsunami is huge as well (200 km)
    - Dynamics of tsunamis are "essentially" governed by the shallow water equations.
    - Phase speed of propagation $v_\phi \approx \sqrt{gH}$ (H ocean depth)
    - Use λ instead of L in the derivation $\rightarrow$ shallow water models : justify the use of Saint-Venant equations for some tsunamis.

Saint-Venant equations for certains tsunamis ? ? ?

- Tsunamis are water waves that start in the deep ocean : H is huge
- But, the wavelength λ of the tsunami is huge as well (200 km)
- Tsunami runup onto a complex three dimensional Monai Valley :

	Adap. sim.	Unif. sim.
T_f	30 s	30 s
Nb. blocks	240	240
Nb. cells	8 000-40 000	62 000
Re-mesh. δt	0.25 s	X
CFL	0.5	0.5

Table – Numerical parameters



Numerical water height (coloration is issue from the kinetic energy) at $t = 11.25$ s

[BEG12] K. Pons, **M. Ersoy**.
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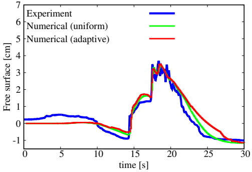
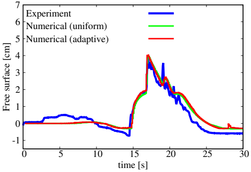
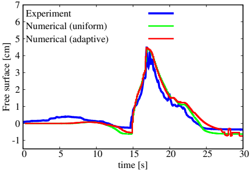


(a) Gauge 1



(b) Gauge 2



(c) Gauge 3

FIGURE – Free surface results at different positions : experimental data versus numerical simulation with and without mesh adaptivity

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  - Land-slide/subaerial landslide generated tsunamis (depending on landslide thickness, water depth) cannot be represented by hydrostatic models ! $\rightarrow$ *Glimsdal, Pedersen, Harbitz, Lovholt, Dutykh, Bonneton*, *etc.*

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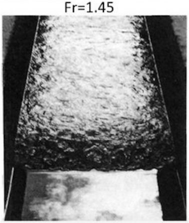

"Strong" bore

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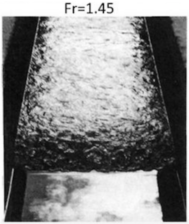

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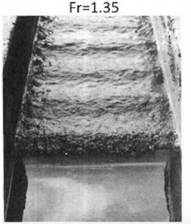

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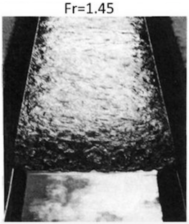

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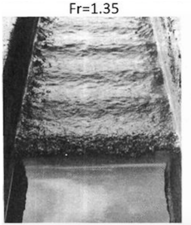

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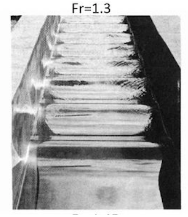

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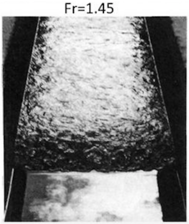

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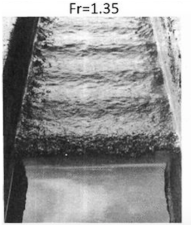

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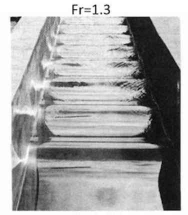

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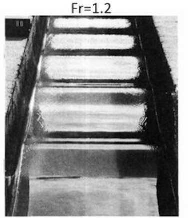

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- Of course, Navier-Stokes equation can deal for both but too costly !

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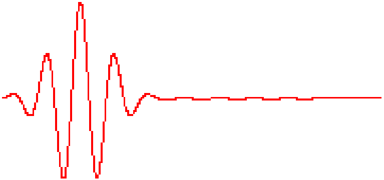
Let $\omega = \dfrac{2\pi}{T}$ be the angular frequency (pulsation) and $k = \dfrac{2\pi}{\lambda}$ wavenumber.

- A wave $\phi(kx - \omega t)$ is characterised by two different characteristic speeds
  - phase velocity $C_p = \dfrac{\omega}{k}$ which corresponds to the displacement of the wave fronts
  - group velocity $C_g = \dfrac{\partial \omega}{\partial k}$ which corresponds to the displacement of the wave's envelope
  - dispersion relation is given by $\omega = C_p k$
- If C_p is constant then the wave is not dispersive.



Dispersive wave

Non dispersive wave

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- If C_p is constant then the wave is not dispersive.
- According to linear Stokes'theory, noting H the depth, the dispersion relation is

$$\underline{\omega^2 = gk\tanh(kH)}$$

Formally, $\dfrac{H}{\lambda} \ll 1$,

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- at order > 1, $\left(\dfrac{\omega}{k}\right)^2 \approx gH - gk^2H^3 + \ldots \rightsquigarrow$ Dispersive models

- Everything starts with Russell's "Wave of translation"

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion ; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation". **John** Scott Russell

- Everything starts with Russell's "Wave of translation"
- Proof of the stability of the solitary wave given by Boussinesq (1872)/Korteweg and Gustav de Vries (1895) through a 1D scalar equation : a perfect equilibrium between non-linearities and the dispersive terms

$$u_t + 6uu_x + u_{xxx} = 0$$

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- On the basis of this work, several models have been proposed :
  - **1967** : a first 2D formulation for non flat weakly dispersive and weakly non linear model of Boussinesq type was proposed by Peregrine.

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  - **1953** : A first 1D fully non-linear ($\varepsilon = O(1)$) and weakly dispersive equation for flat bottom was derived by Serre motivated by the fact that wave dynamics is strongly nonlinear close to shoaling zone.

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  - **1976** : Green and Naghdi derived the famous 2D fully nonlinear dispersive equations for uneven bottom (1D below)

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial t}h + \dfrac{\partial}{\partial x}(hu) = 0\\ \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}\left(hu^2 + \dfrac{h^2}{2F_r^2}\right) + \mu\dfrac{\partial}{\partial x}\left(\dfrac{h^3}{3}\mathcal{D}(u)\right) = 0\end{array}\right. \quad \text{with}$$

$$\mathcal{D}(u) = \left(\frac{\partial}{\partial x}u\right)^2 - \frac{\partial}{\partial t}\frac{\partial}{\partial x}u - u\frac{\partial}{\partial x}\frac{\partial}{\partial x}u$$

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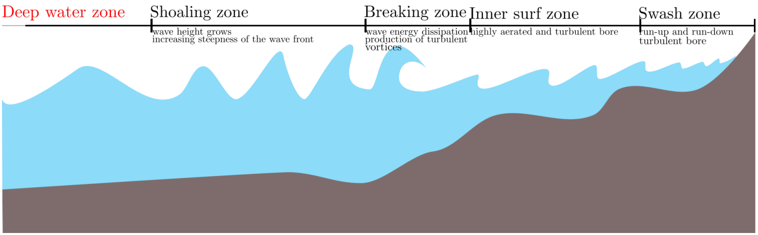
  - 1976 : Green and Naghdi derived the famous 2D fully nonlinear dispersive equations for uneven bottom.
  - Nowadays : Marche, Lannes, Bonneton, Durand, Cienfuegos, Dutykh, Gavrilyuk, Richard, Sainte-Marie, . . . proposed several improvements.

- SGN based models are certainly the most appropriate ones for dispersive waves. [a]

a. Lannes, Marche, Durand, Bonneton, Cienfuegos, Dutykh, Gavrilyuk,. . .

- SGN based models are certainly the most appropriate ones for dispersive waves.
- But, dispersive and non dispersive waves can coexist during the Tsunami's life ...
  - Deep water zone : Depth-averaged models hydrostatic and non-hydrostatic models are valid but dispersive codes boosts the CPU times and memory requirements



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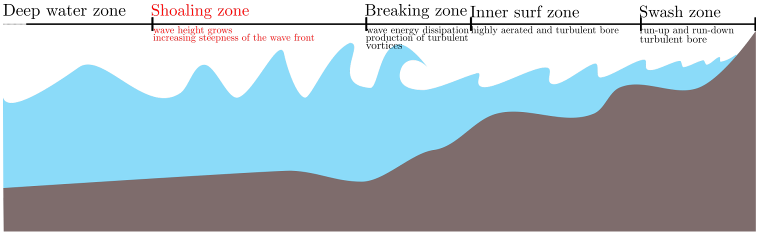
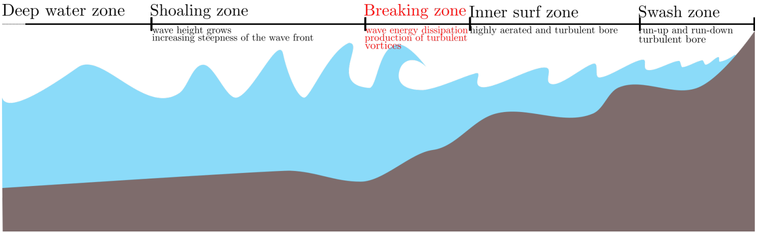
  - Shoaling zone : hydrostatic models are (often) not valid in this zone, leading to an incorrect growth of the wave, yielding to an incorrect prediction of the location of wave breaking



Coming back to Tsunami propagation : toward a new non-hydrostatic model

- SGN based models are certainly the most appropriate ones for dispersive waves.
- But, dispersive and non dispersive waves can coexist during the Tsunami's life . . .
  - Breaking zone : hydrostatic models (SVE) can accurately reproduce broken wave dissipation and swash oscillations without any ad-hoc parametrisation



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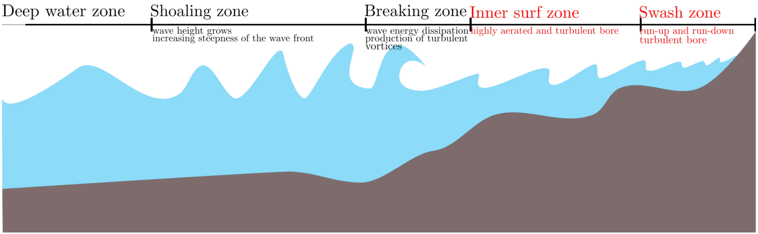
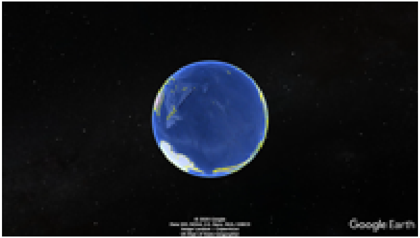
  - Inner surf and swash zones : predominant non-linearities (SVE)



- SGN based models are certainly the most appropriate ones for dispersive waves.
- But, dispersive and non dispersive waves can coexist during the Tsunami's life ...
- Dissipative models are required [a] : "switching from one model to an other"

a. Lannes, Marche, Durand, Bonneton, Cienfuegos, Dutykh, Gavrilyuk, Pons, ...

- Waves may penetrate through rivers/channel much faster inland than the coastal inundation reaches over the ground, and may lead flooding in low-lying areas located several km away from the coastline !



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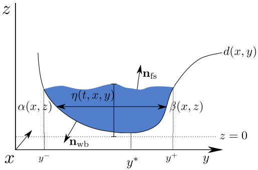
  - 2D models for rivers/channels can be used but costly in the large scale simulation
  - Hydrostatic 1D section-averaged models are well-mastered
  - Non-hydrostatic 1D section-averaged have not yet been derived
    $\rightarrow$ toward the first full non-linear and weakly dispersive section-averaged model

Incompressible Euler equations

$$\begin{aligned} \operatorname{div}(\rho_0 \boldsymbol{u}) &= 0, \\ \frac{\partial}{\partial t}(\rho_0 \boldsymbol{u}) + \operatorname{div}(\rho_0 \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \rho_0 \boldsymbol{F} &= 0 \end{aligned}$$



with

$\boldsymbol{u} = (u, v, w)$	:	velocity field
ρ_0	:	density
$\boldsymbol{F} = (0, 0, -g)$	:	external force
p	:	pressure

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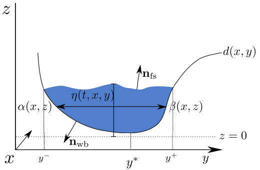

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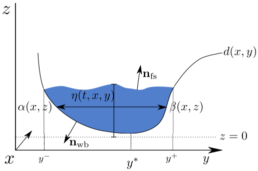
completed with the irrotational relations

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \ .$$

Incompressible and irrotational Euler equations



$$
\begin{array}{rcl}
\mathrm{div}(\rho_0 \boldsymbol{u}) & = & 0, \\
\dfrac{\partial}{\partial t}(\rho_0 \boldsymbol{u}) + \mathrm{div}(\rho_0 \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \rho_0 \boldsymbol{F} & = & 0
\end{array}
$$

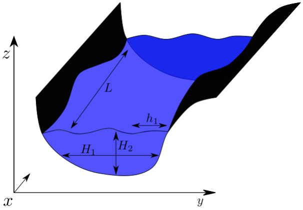
- free surface kinematic boundary condition,

$$\boldsymbol{u}\cdot\boldsymbol{n}_{\mathrm{fs}} = \frac{\partial}{\partial t}\boldsymbol{m}\cdot\boldsymbol{n}_{\mathrm{fs}} \text{ and } p(t,\boldsymbol{m}) = p_0,\ \forall \boldsymbol{m}(t,x,y) = (x,y,\eta(t,x,y)) \in \Gamma_{\mathrm{fs}}(t,x)$$

- no-penetration condition on the wet boundary

$$\boldsymbol{u}\cdot\boldsymbol{n}_{\mathrm{wb}} = 0,\ \forall \boldsymbol{m}(x,y) = (x,y,d(x,y)) \in \Gamma_{\mathrm{wb}}(x)$$

Let us define the dispersive parameters



- $\mu_1 = \dfrac{h_1^2}{L^2}$
- $\mu_2 = \dfrac{H_2^2}{L^2}$,

such that

$$h_1 < H_1 = H_2 \ll L, \text{i.e. } \mu_1 < \mu_2^2$$

where

H_1	:	characteristic scale of channel width
h_1	:	characteristic wave-length in the transversal direction
H_2	:	characteristic water depth
$F_r = \dfrac{U}{\sqrt{gH_2}}$	:	Froude's number
$T = \dfrac{L}{U}$	:	characteristic time
$\mathcal{P} = U^2$	:	characteristic pressure
X	:	characteristic length of x

Then, define the dimensionless variables

$$
\begin{array}{lll}
\widetilde{x} = \dfrac{x}{L}, & \widetilde{P} = \dfrac{P}{\mathcal{P}}, & \widetilde{\varphi} = \dfrac{\varphi}{h_1}, \\[2ex]
\widetilde{y} = \dfrac{y}{h_1}, & \widetilde{u} = \dfrac{u}{U}, & \widetilde{d} = \dfrac{d}{H_2}, \\[2ex]
\widetilde{z} = \dfrac{z}{H_2}, & \widetilde{v} = \dfrac{v}{V} = \dfrac{v}{\sqrt{\mu_1}U}, & \widetilde{\eta} = \dfrac{\eta}{H_2} \ . \\[2ex]
\widetilde{t} = \dfrac{t}{T}, & \widetilde{w} = \dfrac{w}{W} \ = \dfrac{w}{\sqrt{\mu_2}U} \ . &
\end{array}
$$

We get

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0\\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} &= 0\\
\mu_1\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial P}{\partial y} &= 0\\
\mu_2\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \frac{\partial P}{\partial z} &= -\frac{1}{{F_r}^2}
\end{aligned}
$$

and

$$
\frac{\partial u}{\partial y} = \mu_1\frac{\partial v}{\partial x},\ \ \mu_1\frac{\partial v}{\partial z} = \mu_2\frac{\partial w}{\partial y},\ \ \frac{\partial u}{\partial z} = \mu_2\frac{\partial w}{\partial x}\ .
$$

Setting $\mu_2 = 0 \Rightarrow \mu_1 = 0$, yields to the hydrostatic pressure

$$\frac{\partial P}{\partial y} = 0$$
$$\frac{\partial P}{\partial z} = -\frac{1}{{F_r}^2}$$

Setting μ_2 small enough and μ_1 negligible, yields to the non hydrostatic pressure

$$\frac{\partial P}{\partial y} = O(\mu_1)$$

$$\frac{\partial P}{\partial z} = -\frac{1}{{F_r}^2} - \mu_2 \frac{d}{dt} w$$

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with

$$P(t,x,y,z) = \frac{\eta(t,x,y) - z}{F_r^2} + \underbrace{\int_z^\eta \mu_2 \frac{d}{dt} w dz}_{P_{\mathrm{nh}}} \ .$$

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$$P(t,x,y,z) = \frac{\eta(t,x,y) - z}{F_r^2} + \underbrace{\int_z^\eta \mu_2 \frac{d}{dt} w dz}_{P_{\mathrm{nh}}} .$$

Since $\dfrac{\partial P}{\partial y} = O(\mu_1)$, we get

$$O(\mu_1) = \frac{\partial P}{\partial y} = \frac{\partial \eta}{\partial y} \underbrace{\left(\frac{1}{F_r^2} + \mu_2 \frac{d}{dt} w dz \right)}_{{\frac{\partial P}{\partial z}}_{|z=\eta}{}^{\neq 0}} \Rightarrow \frac{\partial \eta}{\partial y} = O(\mu_1)$$

Remark I : why $\mu_1 \neq \mu_2$?

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  $\Rightarrow w(t,x,z) = -\left(\int_d^z u_{|z=d}(t,x)\,dz\right)_x + O(\mu)$
- $\Rightarrow u(t,x,z) = f_1(\bar{u}(t,x)) + \mu f_2(z, \bar{u}(t,x), d(x)) + O(\mu^2)$ where $\bar{u}(t,x) = f_3(u_{|z=d}) \ldots$

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Therefore, we assume $\mu_1 < \mu_2$.

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  - Euler equations + boundary conditions :

$$\int_{\partial\Omega(t,x)} \left(\frac{\partial}{\partial t}\boldsymbol{m} + u\frac{\partial}{\partial x}\boldsymbol{m} - \boldsymbol{v} \right) \cdot \boldsymbol{n}\; ds = 0$$

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  - Introduce wet region indicator function Φ which satisfies

$$\frac{\partial}{\partial t}\Phi + \frac{\partial}{\partial x}(\Phi u) + \mathrm{div}_{y,z}\left[\Phi \boldsymbol{v}\right] = 0 \text{ on } \Omega(t) = \bigcup_{0\le x\le 1} \Omega(t,x)$$

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    where $\boldsymbol{v} = (v, w)$.
  - Section-average equations using the approximation

$$\begin{array}{rcl} u(t,x,y,z) & = & \bar{u}(t,x) + \mu_2 B_0(\bar{u}, x, z) + O(\mu_2^2) \\ \eta(t,x,y) & = & \bar{\eta}(t,x) + O(\mu_1) \\ P(t,x,y,z) & = & P_{\mathrm{h}}(t,x,z) + \mu_2 P_{\mathrm{nh}}(t,x,z) + O(\mu_2^2) \end{array}$$

The new model : generalization of the SGN and Free surface flows equations

$$\begin{cases} \dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}Q = 0 \\ \dfrac{\partial}{\partial t}Q + \dfrac{\partial}{\partial x}\left(\dfrac{Q^2}{A} + I_1(x,A)\right) + \mu_2\dfrac{\partial}{\partial x}(\mathcal{D}(u)G(A,x)) = I_2(x,A) \\ +\mu_2\mathcal{G}(u,S,\sigma) + O(\mu_2^2) \end{cases}$$

where

$A = \displaystyle\int_{\Omega(t,x)} dy\,dz$	:	wet area
$Q = A(t,x)u(t,x)$	:	discharge
$I_1 = \displaystyle\int_{\Omega(t,x)} \frac{\eta(t,x)-z}{F_r^2}\sigma(x,z)\,dy\,dz$	:	hydro. press.
$I_2 = -\displaystyle\int_{y^-(t,x)}^{y^+(t,x)} \frac{h(t,x)}{{F_r}^2}\frac{\partial}{\partial x}d(x,y)\,dy$	:	hydro. press. source

- Debyaoui, Ersoy. Asymptotic Analysis, 2020

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial t}A+\dfrac{\partial}{\partial x}Q=0\\ \dfrac{\partial}{\partial t}Q+\dfrac{\partial}{\partial x}\left(\dfrac{Q^2}{A}+I_1(x,A)\right)+\mu_2\dfrac{\partial}{\partial x}(\mathcal{D}(u)G(A,x))=I_2(x,A)\\ +\mu_2\mathcal{G}(u,S,\sigma)+O(\mu_2^2)\end{array}\right.$$

where

$$\mathcal{D}(u)=\left(\frac{\partial}{\partial x}u\right)^2-\frac{\partial}{\partial t}\frac{\partial}{\partial x}u-u\frac{\partial}{\partial x}\frac{\partial}{\partial x}u$$

and

$$G(A,x)=\int_{d^*(x)}^{\eta}\sigma(x,z)\int_z^{\eta}\frac{S(x,s)}{\sigma(x,s)}\,ds\,dz$$

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where

$$\begin{aligned}\mathcal{G}(u,S,\sigma) &= \int_z^\eta \frac{u^2}{\sigma(x,s)}\left(\frac{\dfrac{\partial}{\partial x}S(x,s)\dfrac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)}-\frac{\partial}{\partial x}\frac{\partial}{\partial x}S(x,s)\right)\\ &\quad+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\frac{S(x,s)\dfrac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)^2}\\ &\quad-\left(\frac{\partial}{\partial t}u+u\frac{\partial}{\partial x}u\right)\frac{\dfrac{\partial}{\partial x}S(x,s)}{\sigma(x,s)}\,ds\end{aligned}.$$

$$
\left\{
\begin{array}{l}
\dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}Q = 0 \\
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+\mu_2 \mathcal{G}(u, S, \sigma) + O(\mu_2^2)
\end{array}
\right.
$$

Setting $\sigma = 1$, $d = 1$,

- $A = h$
- $S(x, z) \equiv S(z) \Rightarrow \mathcal{G} = 0$ and $I_2 = 0$
- $G = \dfrac{h^3}{3}$
- $I_1 = \dfrac{h^2}{2F_r^2}$

$$\left\{\begin{array}{l}\frac{\partial}{\partial t}A+\frac{\partial}{\partial x}Q=0\\ \frac{\partial}{\partial t}Q+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}+I_1(x,A)\right)+\mu_2\frac{\partial}{\partial x}(\mathcal{D}(u)G(A,x))=I_2(x,A)\\ +\mu_2\mathcal{G}(u,S,\sigma)+O(\mu_2^2)\end{array}\right.$$

we recover the classical SGN equations on flat bottom

$$\left\{\begin{array}{l}\frac{\partial}{\partial t}h+\frac{\partial}{\partial x}(hu)=0\\ \frac{\partial}{\partial t}(hu)+\frac{\partial}{\partial x}\left(hu^2+\frac{h^2}{2F_r^2}\right)+\mu_2\frac{\partial}{\partial x}\left(\frac{h^3}{3}\mathcal{D}(u)\right)=O(\mu_2^2)\end{array}\right.$$

where

$$\mathcal{D}(u)=\left(\frac{\partial}{\partial x}u\right)^2-\frac{\partial}{\partial t}\frac{\partial}{\partial x}u-u\frac{\partial}{\partial x}\frac{\partial}{\partial x}u$$

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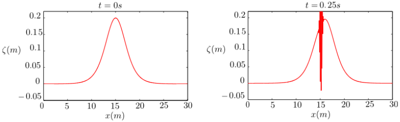
REMARK

Dispersive equation are usually characterised by third order term

$\Downarrow$

time step restriction and may create high frequencies instabilities



Bourdarias, Gerbi, and Ralph Lteif. Computers & Fluids, 156 :283–304, 2017.

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators

$$\mathcal{T}[A,d,\sigma,z](u) = \frac{\partial}{\partial x}(u)\int_z^\eta \frac{S(x,s)}{\sigma(x,s)}\,ds + u\int_z^\eta \frac{1}{\sigma(x,s)}\frac{\partial}{\partial x}S(x,s)\,ds\ ,$$

and

$$\begin{aligned}\mathcal{G}[A,d,\sigma,z](u) \quad = \quad & \int_z^\eta 2\left(\frac{\partial}{\partial x}u\right)^2\frac{S(x,s)}{\sigma(x,s)}+ \\ & \frac{u^2}{\sigma(x,s)}\left(\frac{\frac{\partial}{\partial x}S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)} - \frac{\partial}{\partial x}\frac{\partial}{\partial x}S(x,s)\right) \\ & +\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\frac{S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)^2}\,ds\end{aligned}$$

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
- Define the averaged linear $\overline{\mathcal{T}}$ and the quadratic $\overline{\mathcal{Q}}$ operators

$$\overline{\mathcal{T}}[A,d,\sigma](u,\psi) \quad = \quad \int_{d^*(x)}^{\eta} \psi \mathcal{T}[A,d,\sigma,z](u)\ dz$$

and

$$\overline{\mathcal{G}}[A,d,\sigma](u,\psi) \quad = \quad \int_{d^*(x)}^{\eta} \psi \mathcal{G}[A,d,\sigma,z](u)\ dz$$

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
- Define the averaged linear $\overline{\mathcal{T}}$ and the quadratic $\overline{\mathcal{Q}}$ operators
- Define the operators $\mathcal{L}$ and $\mathcal{Q}$

$$\mathbb{L}[A,d,\sigma](u) \quad = \quad A\mathcal{L}[A,d,\sigma]\left(\frac{u}{A}\right)$$

and

$$\mathcal{Q}[A,d,\sigma](u) \quad = \quad \frac{1}{A}\left[\frac{\partial}{\partial x}\left(\overline{\mathcal{G}}[A,d,\sigma]\,(u,\sigma)\right) - \overline{\mathcal{G}}[A,d,\sigma]\left(u, \frac{\partial}{\partial x}\sigma\right)\right]$$

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
- Define the averaged linear $\overline{\mathcal{T}}$ and the quadratic $\overline{\mathcal{Q}}$ operators
- Define the operators $\mathcal{L}$ and $\mathcal{Q}$
- and finally the operator $\mathbb{L}$

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- Reformulated model

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}(Au) = 0 \\ \left(I_d - \mu_2 \mathbb{L}[A,d,\sigma]\right)\left(\dfrac{\partial}{\partial t}(Au) + \dfrac{\partial}{\partial x}(Au^2)\right) + \dfrac{\partial}{\partial x}I_1(x,A) \\ +\mu_2 A\mathcal{Q}[A,d,\sigma](u) = I_2(x,A) + O(\mu_2^2) \end{array}\right.$$

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Remark

Inverting $I_d-\mu_2\mathbb{L}[A,d,\sigma]$ $\Rightarrow$ no third order term $\Rightarrow$ more stable formulation

- Bonneton, Barthélemy, Chazel, Cienfuegos, Lannes, Marche, and Tissier. European Journal of Mechanics-B/Fluids, 2011
- Debyaoui, Ersoy. Recent Advances in Numerical Methods for Hyperbolic PDE Systems. SEMA SIMAI Springer Series, 2021

- Define the linear $\mathcal{T}$ and the quadratic $\mathcal{Q}$ operators
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- Define the operators $\mathcal{L}$ and $\mathcal{Q}$
- and finally the operator $\mathbb{L}$
- Reformulated model

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}(Au) = 0 \\ (I_d - \mu_2\kappa\mathbb{L}[A,d,\sigma])\left(\dfrac{\partial}{\partial t}(Au) + \dfrac{\partial}{\partial x}(Au^2) + \dfrac{\kappa-1}{\kappa}\left(\dfrac{\partial}{\partial x}I_1 - I_2\right)\right) \\ +\dfrac{1}{\kappa}\left(\dfrac{\partial}{\partial x}I_1 - I_2\right) + \mu_2 A\mathcal{Q}[A,d,\sigma](u) = O(\mu_2^2) \end{array}\right.$$

REMARK

A consistent one-parameter $\kappa > 0$ family (up to order $O(\mu_2^2)$) can be introduced to improve the frequency dispersion.

- Bonneton, Barthélemy, Chazel, Cienfuegos, Lannes, Marche, and Tissier. European Journal of Mechanics-B/Fluids, 2011
- Debyaoui, Ersoy. Recent Advances in Numerical Methods for Hyperbolic PDE Systems. SEMA SIMAI Springer Series, 2021

Theorem

Let α,β and $d \in C_b^\infty$ and $A \in W^{1,\infty}(\mathbb{R})$ such that $\inf_{x\in\mathbb{R}} A \geq A_0 > 0$. Then the operator

$$\mathbb{T} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$$

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is well-defined, one-to-one and onto.

- Let $\mu_2 \in (0, 1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$ the space $H^1(\mathbb{R})$ endowed with the norm

$$\| u \|_{\mu_2}^2 = \| u \|_2^2 + \mu_2 \| u_x \|_2^2$$

THEOREM

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- Let $\mu_2 \in (0, 1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$
- Define the bilinear form $a(u, v)$

$$a(u, v) = (A\mathbb{T}u, v) = (Au, v)+$$

$$\mu_2 \left(A \left(\frac{A}{\sqrt{3}u_x} - \frac{\sqrt{3}}{2} d_x u \right), \left(\frac{A}{\sqrt{3}v_x} - \frac{\sqrt{3}}{2} d_x v \right) \right) + (Ad_x u, d_x v)$$

Theorem

Let α,β and $d \in C_b^\infty$ and $A \in W^{1,\infty}(\mathbb{R})$ such that $\inf\limits_{x\in\mathbb{R}} A \geq A_0 > 0$. Then the operator

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is well-defined, one-to-one and onto.

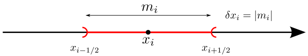
- Let $\mu_2 \in (0, 1)$. Define the space $H^1_{\mu_2}(\mathbb{R})$
- Define the bilinear form $a(u, v)$
- Lax-Milgram theorem

$$\exists!\ u \in H^1_{\mu_2}(\mathbb{R})\ ;\ a(u, v) = (f, v),\ \forall v \in H^1_{\mu_2}(\mathbb{R}),\ f \in L^2(\mathbb{R})$$

$$\Downarrow$$

$$\exists!\ u \in H^1_{\mu_2}(\mathbb{R})\ ;\ \mathbb{T}u = f$$

- From definition of $\mathbb{T}$, we get $u_{xx} = g(A, u, d, \sigma) \in L^2(\mathbb{R}) \Rightarrow u \in H^2(\mathbb{R})$.



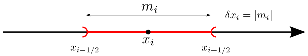
We consider a classical Finite Volume scheme, $\boldsymbol{U} = (A, Q)$

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\delta t^n}{\delta x}\left(\boldsymbol{F}_{i+1/2}(\boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) - \boldsymbol{F}_{i-1/2}(\boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n)\right)$$

where $\boldsymbol{F}_{i\pm1/2} \approx \dfrac{1}{\delta t^n}\displaystyle\int_{m_i} \boldsymbol{F}(\boldsymbol{U}(t, x_{i+1/2}))\, dx$ is a Finite volume solver,

with

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} Au \\ Au^2 + \dfrac{\kappa - 1}{\kappa}\left(I_1 \qquad\qquad \right) \end{pmatrix}$$



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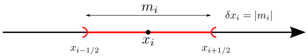
where $\boldsymbol{F}_{i\pm1/2} \approx \dfrac{1}{\delta t^n}\displaystyle\int_{m_i} \boldsymbol{F}(\boldsymbol{U}(t, x_{i+1/2}))\, dx$ is a Finite volume solver, for instance, with upwind technique to deal with source term

$$\boldsymbol{F}_{i\pm1/2} = \frac{\boldsymbol{F}(\boldsymbol{U}) + \boldsymbol{F}(\boldsymbol{V})}{2} - \frac{s_i^n}{2}(\boldsymbol{V} - \boldsymbol{U})$$

with

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} Au \\ Au^2 + \dfrac{\kappa - 1}{\kappa}\left(I_1 -'' \displaystyle\int I_2''\right) \end{pmatrix}$$

- Bourdarias, Ersoy, Gerbi. Journal of Scientific Computing, 2011



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$$-\frac{\delta t^n}{\delta x}\left(\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1} \boldsymbol{D}^n\right)_i$$

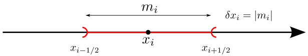
with

$$(\boldsymbol{D}^n)_i = \boldsymbol{D}_{i+1/2}(\boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) - \boldsymbol{D}_{i-1/2}(\boldsymbol{U}_{i-2}^n, \boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n)$$

where $\boldsymbol{D}_{i\pm 1/2}$ and $\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1}$ are the centred approximation of

$$\mathcal{D} = \frac{1}{\kappa}\left(\frac{\partial}{\partial x} I_1 - I_2\right) + \mu_2 A \mathcal{Q} \text{ and } \left[(I_d - \mu_2 \mathbb{L})\right]^{-1}$$

Numerical scheme :



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$$-\frac{\delta t^n}{\delta x}\left(\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1} \boldsymbol{D}^n\right)_i$$

Theorem

The numerical scheme is stable under the classical CFL condition,

$$\max_{\lambda \in \mathrm{Sp}(D_{\boldsymbol{U}} \boldsymbol{F}(\boldsymbol{U}))} |\lambda| \frac{\delta t^n}{\delta x} \leqslant 1 \ .$$

- Debyaoui, Ersoy. Recent Advances in Numerical Methods for Hyperbolic PDE Systems. SEMA SIMAI Springer Series, 2021

- Comparison with the NLSW and the exact solution

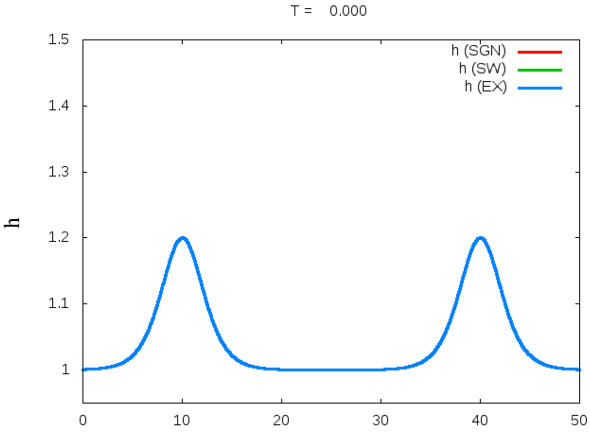
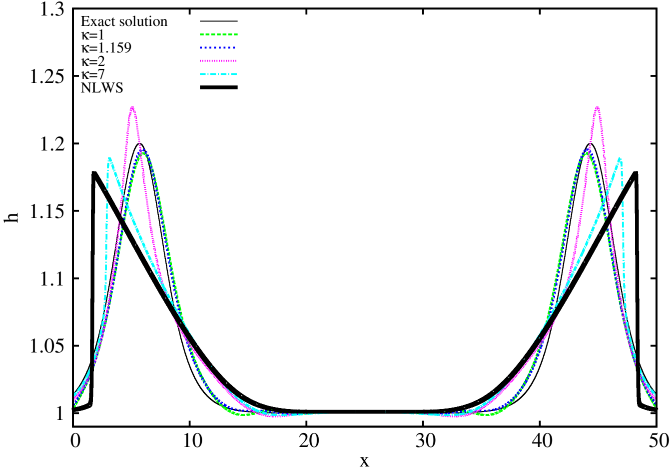


Figure – $\sigma = 1$, $d = 1$, $N = 1000$, $CFL = 0.95$, $T_f = 10$ and $\kappa = 1.159$

- Comparison with the NLSW and the exact solution
- Influence of κ : toward a dissipative shallow water model



(a) Solutions at time $T_f = 10$

Outline

- many things to do (solitary waves, inversion thm, high order numerical scheme, ...)

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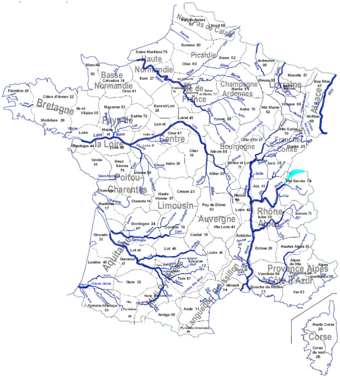
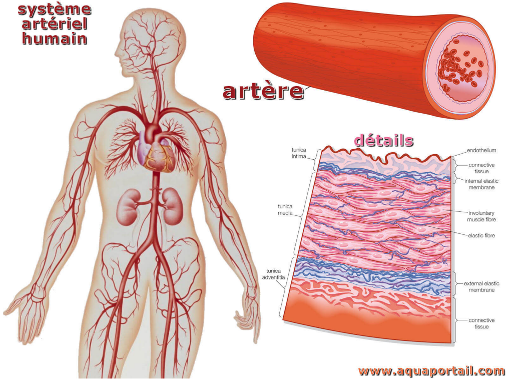
- Human arteries network is rather similar to river/stream one





Thank you

for your

attention