Existence and stability results for some compressible primitive equations

M. Ersoy [1], T. Ngom [2] and M. Sy [3]

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OUTLINE OF THE TALK

1. INTRODUCTION

2. MAIN RESULTS
   - Existence result for the 2D-CPEs
   - A stability result for the 3D-CPEs

3. PERSPECTIVES

Outline

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Context

Navier-Stokes equations (NSEs) or Euler equations (EEs) on
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Hydrostatic approximation (asymptotic analysis with $\varepsilon = H/L = W/V \ll 1$ and rescaling $\tilde{x} = x/L$, $\tilde{y} = y/H$, $\tilde{u} = u/U$ $\tilde{w} = w/W$) $\longrightarrow$ Primitive equations (PEs)

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Geophysical Fluid Dynamics.
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$\downarrow$ [GP]

Averaged PEs with respect to depth or altitude y $\longrightarrow$ Saint-Venant Equations (SVEs)

J. Pedlowski
Geophysical Fluid Dynamics.
2nd Edition, Springer-Verlag, New-York, 1987.

J.-F Gerbeau and B. Perthame
Derivation of viscous Saint-Venant system for laminar shallow water ; numerical validation.
Discrete Contin. Dyn. Syst. Ser. B, 1(1), 2001.

Atmosphere dynamic

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  - Compressible fluid
  - Small vertical extension with respect to horizontal
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  Hydrostatic approximation $\longrightarrow$ Compressible Primitive Equations (CPEs)

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\rho + \rho \mathrm{div}\mathbf{U} & = & 0 \\ \rho \dfrac{d}{dt}\mathbf{u} + \nabla_x p & = & \mathrm{div}_x(\sigma_x) + f \\ \partial_t(\rho v) + \mathrm{div}(\rho \mathbf{U} v) + \partial_y p(\rho) & = & -\rho g + \mathrm{div}_y(\sigma_y) \\ p(\rho) & = & c^2\,\rho \end{array}\right.$$

with $\dfrac{d}{dt} := \partial_t + \mathbf{u}\cdot\nabla_x + v\partial_y$

and σ_{xx} xx component of the viscous stress strensor

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M. Ersoy and T. Ngom
Existence of a global weak solution to one model of Compressible Primitive Equations.
Submitted, 2010.

M. Ersoy, T. Ngom and M. Sy
Compressible primitive equations : formal derivation and stability of weak solutions.
Nonlinearity, 24(1), pp 79-96, 2011.

Main difference with respect to the constant viscous term (classical) found in the literature (see, for instance, R. Temam and M. Ziane *Handbook of mathematical fluid dynamics. Vol. III*, 2004.) : here

viscosities depend on the density and **are anisotropic**.

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A useful change of variables [EN10]

Let us consider the following two dimensional problem :

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\rho + \rho \mathrm{div}\mathbf{U} & = & 0 \\ \rho\dfrac{d}{dt}\mathbf{u} + c^2\partial_x\rho & = & \partial_x(\nu_1(t,x,y)\partial_x u) + \partial_y(\nu_2(t,x,y)\partial_y u) \\ c^2\partial_y\rho = -g\rho & & \end{array}\right. \quad (1)$$

on $\Omega = \{(x,y);\, 0 < x < l,\, 0 < y < h\}$ with :

$$u_{|x=0} = u_{|x=l} = 0, \quad v_{|y=0} = v_{|y=h} = 0, \quad \partial_y u_{|y=0} = \partial_y u_{|y=h} = 0$$

$$u_{|t=0} = u_0(x,y), \quad \rho_{|t=0} = \xi_0(x)e^{-g/c^2 y}$$

where $0 < m \leqslant \xi_0 \leqslant M < \infty$.
and $\mathbf{U} = (\mathbf{u}, v) \in \mathbb{R}^2$
or equivalently, in conservative form :

$$\left\{\begin{array}{rcl} \partial_t\rho + \partial_x(\rho\mathbf{u}) + \partial_y(\rho v) & = & 0 \\ \partial_t(\rho\mathbf{u}) + \partial_x(\rho\mathbf{u}^2) + \partial_y(\rho\mathbf{u}v) + c^2\partial_x\rho & = & \partial_x(\nu_1(t,x,y)\partial_x\mathbf{u}) \\ & & +\partial_y(\nu_2(t,x,y)\partial_y\mathbf{u}) \\ c^2\partial_y\rho & = & -g\rho \end{array}\right.$$

Model formally closed to GK Model : around a useful change of variables . . .

Find a change of variables to get a similar model as in B. V. Gatapov and A. V. Kazhikhov, *Siberian Mathematical Journal*, 46(5), pp 805-812, 2005., i.e.,
using the hydrostatic equation $c^2\partial_y\rho(t,\mathbf{x},y) = -g\,\rho(t,\mathbf{x},y)$
map

$$\rho(t,\mathbf{x},y)\rightarrow\xi(t,\mathbf{x})$$

so-called *stratified property of the density*

A useful change of variables [EN10]

Perform the following steps

$$\begin{cases} \partial_t \rho + \partial_x(\rho \mathbf{u}) + \partial_y(\rho v) & = & 0 \\ \partial_t(\rho \mathbf{u}) + \partial_x(\rho \mathbf{u}^2) + \partial_y(\rho \mathbf{u} v) + c^2 \partial_x \rho & = & \partial_x(\nu_1(t,x,y)\partial_x \mathbf{u}) \\ & & +\partial_y(\nu_2(t,x,y)\partial_y \mathbf{u}) \\ c^2 \partial_y \rho & = & -g\rho \end{cases}$$

Then,

- Set $\rho(t,x,y) = \xi(t,x)e^{-\frac{g}{c^2}y}$, $\nu_1(t,x,y) = \bar{\nu_1} e^{-\frac{g}{c^2}y}$, $\nu_2(t,x,y) = \bar{\nu_2} e^{\frac{g}{c^2}y}$, $(\bar{\nu_1}, \bar{\nu_2}) \in \mathbb{R}^2$ and multiply by $e^{\frac{g}{c^2}y}$

A USEFUL CHANGE OF VARIABLES [EN10]

Perform the following steps

$$\left\{\begin{array}{rcl} \partial_t\xi + \partial_x(\xi\mathbf{u}) + e^{\frac{g}{c^2}y}\partial_y(\xi e^{-\frac{g}{c^2}y}v) & = & 0 \\ \partial_t(\xi\mathbf{u}) + \partial_x(\xi\mathbf{u}^2) + e^{\frac{g}{c^2}y}\partial_y(\xi\mathbf{u}e^{-\frac{g}{c^2}y}v) + c^2\partial_x\xi & = & \overline{\nu_1}\partial_{xx}\mathbf{u} \\ & & +\overline{\nu_2}e^{\frac{g}{c^2}y}\partial_y(e^{\frac{g}{c^2}y}\partial_y\mathbf{u}) \\ e^{\frac{g}{c^2}y}c^2\left(\partial_y(\xi)e^{-\frac{g}{c^2}y} + \xi\partial_y\left(e^{-\frac{g}{c^2}y}\right)\right) = -\xi g & & \end{array}\right.$$

Then,

- Set $\rho(t,x,y) = \xi(t,x)e^{-\frac{g}{c^2}y}$, $\nu_1(t,x,y) = \bar{\nu_1}e^{-\frac{g}{c^2}y}$, $\nu_2(t,x,y) = \bar{\nu_2}e^{\frac{g}{c^2}y}$, $(\bar{\nu_1},\bar{\nu_2}) \in \mathbb{R}^2$ and multiply by $e^{\frac{g}{c^2}y}$

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 & & +\overline{\nu_2} e^{\frac{g}{c^2}y}\partial_y(e^{\frac{g}{c^2}y}\partial_y \mathbf{u}) \\
e^{\frac{g}{c^2}y} c^2 \left(\partial_y(\xi) e^{-\frac{g}{c^2}y} + \xi \partial_y\left(e^{-\frac{g}{c^2}y}\right)\right) = -\xi g & &
\end{array}
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Then,

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- Set $\partial_z \cdot = e^{\frac{g}{c^2}y}\partial_y \cdot$ and $w = e^{-\frac{g}{c^2}y} v$

A useful change of variables [EN10]

Perform the following steps

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- Set $\partial_z\cdot = e^{\frac{g}{c^2}y}\partial_y\cdot$ and $w = e^{-\frac{g}{c^2}y}v$

A useful change of variables [EN10]

Finally, we get :

$$\left\{\begin{array}{rcl} \partial_t \xi + \partial_x(\xi \mathbf{u}) + \partial_z(\xi w) & = & 0 \\ \partial_t(\xi \mathbf{u}) + \partial_x(\xi \mathbf{u}^2) + \partial_z(\xi \mathbf{u} w) + c^2 \partial_x \xi & = & \overline{\nu_1} \partial_{xx} \mathbf{u} + \overline{\nu_2} \partial_{zz} \mathbf{u} \\ \partial_z \xi & = & 0 \end{array}\right.$$

or equivalently, in non-conservative form :

$$\left\{\begin{array}{rcl} \frac{d}{dt}\xi + \xi \mathrm{div} \mathbf{U} & = & 0 \\ \xi \frac{d}{dt}\mathbf{u} + c^2 \partial_x \xi & = & \overline{\nu_1} \partial_{xx} \mathbf{u} + \overline{\nu_2} \partial_{zz} \mathbf{u} \\ \partial_z \xi & = & 0 \end{array}\right.$$

with $\mathbf{U} := (\mathbf{u}, w)$, $\dfrac{D}{Dt} := \partial_t + \mathbf{U} \cdot \nabla$, $\nabla := (\partial_x, \partial_z)^t$, div $:= \partial_x + \partial_z$
and corresponds exactly to the model studied by B. V. Gatapov and A. V. Kazhikhov, *Siberian Mathematical Journal*, 46(5), pp 805-812, 2005. : existence of weak solutions global in time for the model with $(\rho, \mathbf{u})$ is then a straightforward consequence.

Main result

Theorem

Assume that initial data (ξ_0, u_0) *satisfies :*
$(\xi_0, u_0) \in W^{1,2}(\Omega), \quad u_{0|x=0} = u_{0|x=l} = 0.$
Then $\rho(t,x,y)$ *is a bounded strictly positive function and (1)-(2) has a weak solution in the following sense : a weak solution of (1)-(2) is a collection* (ρ, u, v) *of functions such that* $\rho \geqslant 0$ *and*

$$\rho \in L^\infty(0,T; W^{1,2}(\Omega)),\ \partial_t \rho \in L^2(0,T;L^2(\Omega)),$$

$$u \in L^2(0,T;W^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)),\ v \in L^2(0,T;L^2(\Omega))$$

which satisfies (1) in the distribution sense ; in particular, the integral identity holds for all $\phi_{|t=T} = 0$ *with compact support :*

$$\begin{aligned}&\int_0^T \int_\Omega \rho u \partial_t \phi + \rho u^2 \partial_x \phi + \rho u v \partial_z \phi + \rho \partial_x \phi + \rho v \phi \, dxdydt \\ &= - \int_0^T \int_\Omega \bar{\nu}_1 \partial_x u \partial_x \phi + \bar{\nu}_2 \partial_y u \partial_y \phi \, dxdydt + \int_\Omega u_0 \rho_0 \phi_{|t=0} \, dxdy\end{aligned}$$

OUTLINE

The 3D-CPEs

Let us consider the following model on $\Omega = \{(x,y);\, x \in \mathbb{T}^2,\, 0 < y < 1\}$:

$$\left\{\begin{array}{l} \dfrac{d}{dt}\rho + \rho \mathsf{div}\mathbf{U} = 0, \\ \rho \dfrac{d}{dt}\mathbf{u} + \nabla_x p = 2\mathsf{div}_x\left(\nu_1(t,x,y)D_x(\mathbf{u})\right) + \partial_y\left(\nu_2(t,x,y)\partial_y \mathbf{u}\right), \\ \partial_y p = -g\rho, \\ p(\rho) = c^2\,\rho \end{array}\right. \qquad (1)$$

with

$$\begin{array}{l} \text{periodic conditions on } \partial\Omega_x, \\ v_{|y=0} = v_{|y=H} = 0, \\ \partial_y \mathbf{u}_{|y=0} = \partial_y \mathbf{u}_{|y=H} = 0. \end{array}$$

and

$$\begin{array}{l} \mathbf{u}(0,x,y) = \mathbf{u}_0(x,y), \\ \rho(0,x,y) = \xi_0(x)e^{-g/c^2 y} \end{array}$$

where

$$0 \leqslant \xi_0(x) \leqslant M < +\infty.$$

Energy estimates ? ? ?

Let us multiply the previous system by **U**, we get :

$$\frac{d}{dt}\int_\Omega (\rho|\mathbf{u}|^2 + \rho\ln\rho - \rho + 1)\,dxdy + \int_\Omega 2\nu_1|D_x(\mathbf{u})|^2 + \nu_2|\partial_y^2\mathbf{u}|\,dxdy + \int_\Omega \rho g v\,dxdy$$

where $\int_\Omega \rho g v\,dxdy > 0? \quad < 0?$

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Could we simply multiply by **u** instead of **U** ?

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No, we loss information on v.

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where $\displaystyle\int_\Omega \rho g v\,dxdy > 0? \quad < 0?$

Could we simply multiply by **u** instead of **U** ?

No, we loss information on v.

However, if the rhs of the hydrostatic equation is zero, then we obviously get the following relation on the vertical speed

$$\partial_{zz}w = -\frac{1}{\xi}\mathrm{div}_x(\xi\partial_z\mathbf{u})$$

and constitute a crucial information to get additional estimates.

Consequently, we systematically perform the previous change of variables, i.e. changes $(\rho,\mathbf{u},v)$ in $(\xi,\mathbf{u},w)$.

VISCOSITIES ? ? ?

If we choose the previous viscosities, we get :

$$\left\{\begin{array}{l}\dfrac{d}{dt}\xi+\xi\,\mathsf{div}\mathbf{U}=0,\\ \xi\dfrac{d}{dt}\mathbf{u}+\nabla_x p=\overline{\nu_1}\Delta_x\mathbf{u}+\overline{\nu_2}\partial_{yy}\mathbf{u},\\ \partial_z\xi=0\end{array}\right.$$

4.

Viscosities ? ? ?

If we choose the previous viscosities, we get :

$$\left\{ \begin{array}{l} \dfrac{d}{dt}\xi + \xi \mathrm{div}\mathbf{U} = 0, \\ \xi \dfrac{d}{dt}\mathbf{u} + \nabla_x p = \overline{\nu_1}\Delta_x \mathbf{u} + \overline{\nu_2}\partial_{yy}\mathbf{u}, \\ \partial_z \xi = 0 \end{array} \right.$$

- energy estimates OK !

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Viscosities ? ? ?

If we choose the previous viscosities, we get :

$$\begin{cases} \dfrac{d}{dt}\xi + \xi \mathrm{div}\mathbf{U} = 0, \\ \xi \dfrac{d}{dt}\mathbf{u} + \nabla_x p = \overline{\nu_1}\Delta_x \mathbf{u} + \overline{\nu_2}\partial_{yy}\mathbf{u}, \\ \partial_z \xi = 0 \end{cases}$$

- energy estimates OK !
- No way to establish an existence results[4] : Lagrangian coordinates approach as in B. V. Gatapov and A. V. Kazhikhov, *Siberian Mathematical Journal*, 46(5), pp 805-812, 2005. fails.

4. up to our knowledge

Viscosities ? ? ?

Choose $\nu_1(t,x,y) = \bar{\nu}_1\rho(t,x,y)$ and $\nu_2(t,x,y) = \bar{\nu}_2\rho(t,x,y)e^{2y}$
with $\bar{\nu}_i > 0$, we get :

$$\left\{\begin{array}{l} \dfrac{d}{dt}\xi + \xi(\mathrm{div}_x\mathbf{u} + \partial_z w) = 0, \\ \xi\dfrac{d}{dt}\mathbf{u} + c^2\nabla_x\xi = 2\bar{\nu}_1\mathrm{div}_x\left(\xi D_x(\mathbf{u})\right) + \bar{\nu}_2\partial_z\left(\xi\nu_2(t,x,z)\partial_z\mathbf{u}\right), \\ \partial_z\xi = 0, \\ p(\xi) = c^2\,\xi \end{array}\right. \tag{2}$$

Then,

- Existence ? ? ?
- Stability of weak solutions : Yes ! ! ! by adding a regularizing term (combined to viscous term) allows to pass to the limit in the non-linear term $\xi\mathbf{u}\otimes\mathbf{u}$ (BD-entropy).

With these settings

Multiply by **U**, the energy reads :

$$\frac{d}{dt}\int_{\Omega'}\left(\xi\frac{\mathbf{u}^2}{2}+(\xi\ln\xi-\xi+1)\right)dxdz+\int_{\Omega'}\xi(2\bar{\nu}_1|D_x(\mathbf{u})|^2+\bar{\nu}_2|\partial_z\mathbf{u}|^2)\,dxdz$$
$$+r\int_{\Omega'}\xi|\mathbf{u}|^3\,dxdz\leqslant 0 \quad (3)$$

which provides the uniform estimates :

$$\begin{aligned}
\sqrt{\xi}\mathbf{u} \text{ is bounded in } & L^\infty(0,T;(L^2(\Omega^{'}))^2),\\
\xi^{\frac{1}{3}}\mathbf{u} \text{ is bounded in } & L^3(0,T;(L^3(\Omega^{'}))^2),\\
\sqrt{\xi}\partial_z\mathbf{u} \text{ is bounded in } & L^2(0,T;(L^2(\Omega^{'}))^2),\\
\sqrt{\xi}D_x(\mathbf{u}) \text{ is bounded in } & L^2(0,T;(L^2(\Omega^{'}))^{2\times 2}),\\
\xi\ln\xi-\xi+1 \text{ is bounded in } & L^\infty(0,T;L^1(\Omega^{'})).
\end{aligned}$$

With these settings

Following BD the strong convergence of $\sqrt{\xi}\mathbf{u}$ required to pass to the limit in the non linear term $\xi\mathbf{u}\otimes\mathbf{u}$ is obtained by the BD entropy :
Take the gradient of the mass equation, multiply by $2\bar{\nu}_1$, write the term $\nabla_x\xi$ as $\xi\nabla_x\ln\xi$, combine with the momentum equations, to get the entropy inequality :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega'}\left(\xi|\mathbf{u}+2\bar{\nu}_1\nabla_x\ln\xi|^2+2(\xi\log\xi-\xi+1)\right)\,dxdz \\ +\int_{\Omega'}2\bar{\nu}_1\xi|\partial_z w|^2+2\bar{\nu}_1\xi|A_x(u)|^2+\bar{\nu}_2\xi|\partial_z\mathbf{u}|^2\,dxdz \\ +\int_{\Omega'}r\xi|\mathbf{u}|^3+2\bar{\nu}_1 r|\mathbf{u}|\mathbf{u}\nabla_x\xi+8\bar{\nu}_1|\nabla_x\sqrt{\xi}|^2\,dxdz=0.\end{aligned}\tag{4}$$

which gives the following estimates :

$$\begin{aligned}\nabla\sqrt{\xi}\text{ is bounded in }L^\infty(0,T;(L^2(\Omega^{'}))^3),\\ \sqrt{\xi}\partial_z w\text{ is bounded in }L^2(0,T;L^2(\Omega^{'})),\\ \sqrt{\xi}A_x(\mathbf{u})\text{ is bounded in }L^2(0,T;(L^2(\Omega^{'}))^{2\times2})\,.\end{aligned}$$

With these settings

Define the set of function $\rho \in \mathcal{PE}(\mathbf{u}, v; y, \rho_0)$ which satisfy

$$
\begin{array}{ll}
\rho \in L^\infty(0,T;L^3(\Omega)), & \sqrt{\rho} \in L^\infty(0,T;H^1(\Omega)), \\
\sqrt{\rho}\mathbf{u} \in L^2(0,T;(L^2(\Omega))^2), & \sqrt{\rho}v \in L^\infty(0,T;L^2(\Omega)), \\
\sqrt{\rho}D_x(\mathbf{u}) \in L^2(0,T;(L^2(\Omega))^{2\times 2}), & \sqrt{\rho}\partial_y v \in L^2(0,T;L^2(\Omega)), \\
\nabla\sqrt{\rho} \in L^2(0,T;(L^2(\Omega))^3) &
\end{array}
$$

with $\rho \geqslant 0$ and where $(\rho, \sqrt{\rho}\mathbf{u}, \sqrt{\rho}v)$ satisfies :

$$
\left\{
\begin{array}{l}
\partial_t\rho + \mathrm{div}_x(\sqrt{\rho}\sqrt{\rho}\mathbf{u}) + \partial_y(\sqrt{\rho}\sqrt{\rho}v) = 0, \\
\rho_{t=0} = \rho_0.
\end{array}
\right. \tag{5}
$$

With these settings

Define the integral operators, for any smooth test function φ with compact support such as $\varphi(T,x,y)=0$ and $\varphi_0=\varphi_{t=0}$:

$$
\begin{aligned}
\mathcal{A}(\rho,\mathbf{u},v;\varphi,dy) \quad = \quad & -\int_0^T\int_\Omega \rho\mathbf{u}\partial_t\varphi\,dxdydt \\
& +\int_0^T\int_\Omega (2\nu_1(t,x,y)\rho D_x(\mathbf{u})-\rho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi\,dxdydt \\
& +\int_0^T\int_\Omega r\rho|\mathbf{u}|\mathbf{u}\varphi\,dxdydt-\int_0^T\int_\Omega \rho\mathrm{div}(\varphi)\,dxdydt \\
& -\int_0^T\int_\Omega \mathbf{u}\partial_y(\nu_2(t,x,y)\partial_y\varphi)\,dxdydt \\
& -\int_0^T\int_\Omega \rho v\mathbf{u}\partial_y\varphi\,dxdydt
\end{aligned}
$$

$$\mathcal{B}(\rho,\mathbf{u},v;\varphi,dy)=\int_0^T\int_\Omega \rho v\varphi\,dxdydt$$

and

$$\mathcal{C}(\rho,\mathbf{u};\varphi,dy)=\int_\Omega \rho_{|t=0}\mathbf{u}_{|t=0}\varphi_0\,dxdy$$

Definition

A weak solution of 3D-CPEs on $[0,T]\times\Omega$, with boundary conditions and initial conditions, is a collection of functions $(\rho,\mathbf{u},v)$ such as $\rho\in\mathcal{PE}(\mathbf{u},v;y,\rho_0)$ and the following equality holds for all smooth test function φ with compact support such as $\varphi(T,x,y)=0$ and $\varphi_0=\varphi_{t=0}$:

$$\mathcal{A}(\rho,\mathbf{u},v;\varphi,dy)+\mathcal{B}(\rho,\mathbf{u},v;\varphi,dy)=\mathcal{C}(\rho,\mathbf{u};\varphi,dy)\ .$$

Theorem

Let $(\rho_n, \mathbf{u}_n, v_n)$ *be a sequence of weak solutions of 3D-CPEs, with boundary conditions and initial conditions, satisfying entropy inequalities* (3) *and* (4) *such as*

$$\rho_n \geqslant 0, \quad \rho_0^n \to \rho_0 \ in\ L^1(\Omega), \quad \rho_0^n \mathbf{u}_0^n \to \rho_0 \mathbf{u}_0 \ in\ L^1(\Omega).$$

Then, up to a subsequence,

- ρ_n *converges strongly in* $\mathcal{C}^0(0, T; L^{3/2}(\Omega))$,
- $\sqrt{\rho_n}\mathbf{u}_n$ *converges strongly in* $L^2(0, T; (L^{3/2}(\Omega))^2)$,
- $\rho_n u_n$ *converges strongly in* $L^1(0, T; (L^1(\Omega))^2)$ *for all* $T > 0$,
- $(\rho_n, \sqrt{\rho_n}\mathbf{u}_n, \sqrt{\rho_n} v_n)$ *converges to a weak solution of* (5),
- $(\rho_n, \mathbf{u}_n, v_n)$ *satisfies the energy inequality* (3), *the entropy inequality* (4) *and converges to a weak solution of* (1).

Main steps of the proof

To show the compactness of sequences $(\xi_n, \mathbf{u}_n, w_n)$ in appropriate space function we follow the work of Mellet *et al.* [MV07] :

1. show the strong convergence of the sequence $\sqrt{\xi_n}$,
2. we seek bounds of $\sqrt{\xi_n}\mathbf{u}_n$ and $\sqrt{\xi_n}w_n$,
3. prove the weak convergence of $\xi_n\mathbf{u}_n$,
4. prove the convergence of $\sqrt{\xi_n}\mathbf{u}_n$.

which ends the proof.

A. Mellet and A. Vasseur
On the barotropic compressible Navier-Stokes equations.
Comm. Partial Differential Equations, 32(1-3), pp 431–452, 2007.

Outline

1 Introduction

2 Main results
- Existence result for the 2D-CPEs
- A stability result for the 3D-CPEs

3 Perspectives

Open problems

Several aspects on the well-posedness of these equations are still open.

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- A Challenging Mathematical problem :
  At least in a "thin-layer" domain, could we expect the well-posedness of the compressible Navier-Stokes equations with the equation of state $p(\rho) = \rho$ using the results obtained for the 2D-CPEs ?

Open problems

Several aspects on the well-posedness of these equations are still open :

- With the obtained estimates for the 3D-CPEs, could we construct an approximate sequence of solutions ?
- Unicity for the 2D problem ?
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  At least in a "thin-layer" domain, could we expect the well-posedness of the compressible Navier-Stokes equations with the equation of state $p(\rho) = \rho$ using the results obtained for the 2D-CPEs ?

Thank you

for your

attention

One more thing

Main steps :

Equations are

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\begin{cases}
\rho \dfrac{d}{dt}\mathbf{u} + \nabla_x p = \mu \Delta_x \mathbf{u} + \nu \partial_y^2 \mathbf{u}, \\
\partial_y p = -g\rho, \quad p = c^2\,\rho \\
\dfrac{d}{dt}\rho + \rho \mathrm{div}\mathbf{U} = 0, \\
c_p \dfrac{D}{Dt}\mathcal{T} - \dfrac{1}{\rho}\dfrac{D}{Dt}p = Q_{\mathcal{T}}, \\
\dfrac{D}{Dt} q = Q_q
\end{cases}
$$

$\mu \neq \nu$	constant viscosities
$\mathcal{T}$	Temperature
q	amount of water in air
$Q_{\mathcal{T}}$	heat diffusion from sun
Q_q	molecular diffusion
$\dfrac{d}{dt} =$	$\partial_t + \mathbf{u}\cdot\nabla_x + v\partial_y$
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- Conclusion with Leray's results.

return

Main steps : 2D-CPEs

Equations are

$$\left\{\begin{array}{l} \dfrac{d}{dt}\xi + \xi(\partial_x \mathbf{u} + \partial_z w) = 0, \\ \rho\dfrac{d}{dt}\mathbf{u} + \partial_x \xi = \Delta \mathbf{u}, \\ \partial_z \xi = 0. \end{array}\right. \qquad \text{with } \frac{d}{dt} := \partial_t + \mathbf{u}\cdot\nabla_x + v\partial_z$$

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- *a priori* estimates : $\dfrac{d}{dt}\displaystyle\int_D \xi\frac{\mathbf{u}^2}{2} + \xi\ln\xi - \xi + 1\,dxdz + \int_D (\nabla\mathbf{u})^2\ dxdz$
- Write mean equations in Lagrangian coordinates : $\tau = t$ and $\eta = \displaystyle\int_0^x \xi(t,s)\,ds$

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- Write mean equations in Lagrangian coordinates : $\tau = t$ and $\eta = \displaystyle\int_0^x \xi(t,s)\, ds$
- Show by standard argument (Gronwall inequality, Cauchy-Schwartz,...) that the density is bounded from below and above
- Write mean-oscillation equations and apply a Schauder fixed point theorem

return

Notations

- $x = (x_1, x_2)$ horizontal and y vertical coordinate,
- $\mathbf{U} = (\mathbf{u} = (u_1, u_2), v)$ velocity vector (horizontal and vertical component),
- ρ density,
- p barotropic pressure,
- g gravity constant,
- c^2 usually set to $\mathcal{R}\mathcal{T}$ where $\mathcal{R}$ is the specific gas constant for the air and $\mathcal{T}$ the temperature,
- $\mathrm{div}_x := \partial_{x_1} + \partial_{x_2}$, $D_x = (\nabla_x + \nabla_x^t)/2$,
- $\nu_1(t,x,y) \neq \nu_2(t,x,y)$ represent the anisotropic pair of viscosity depending on the density ρ,
- $\dfrac{D}{Dt} := \partial_t + \mathbf{U} \cdot \nabla$,
- $\dfrac{d}{dt} := \partial_t + \mathbf{u} \cdot \nabla_x + v\partial_y$,
- $2D_x(\mathbf{u}) = \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} = \left(\partial_{x_i} \mathbf{u}_j + \partial_{x_j} \mathbf{u}_i\right)_{1 \leqslant i,j \leqslant 2}$.