A kinetic scheme for pressurized flows in non uniform pipes

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Study of unsteady mixed flows in closed pipes : it may happen that some parts are free surface (FS) and other parts are pressurized (PF) $\rightarrow$ transition phenomenon

induced by sudden changes in the boundary conditions

- failure of a pumping
- rapid change of the discharge
- ...

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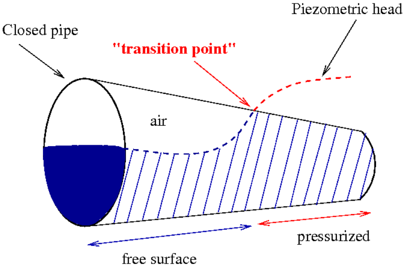
- Free surface area : only a part of the section is filled. Incompressible fluid . . .
- Pressurized area : the section is full-filled. Compressible fluid . . .



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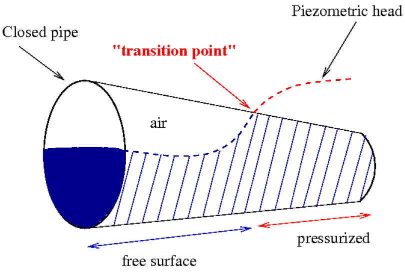

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Towards the simulation of the pressurized flows

- Allievi equations, traditionnally . . .

  But not well adapted . . .
- The artifice of Preissman slot (Cunge et Wegner 1965)

  But depressurisation phenomenon : seen like free surface transition
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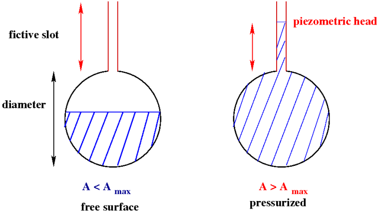

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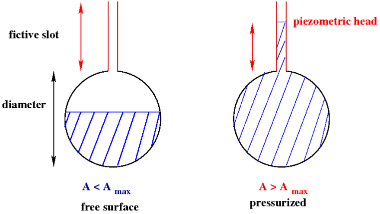

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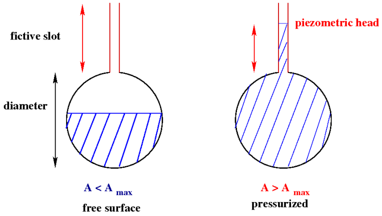

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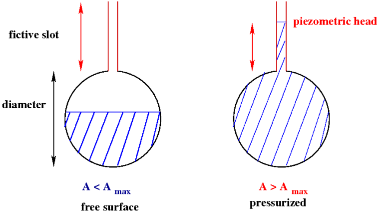

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Formal derivation of 3D Euler compressible

How take into account the variable section in the model?

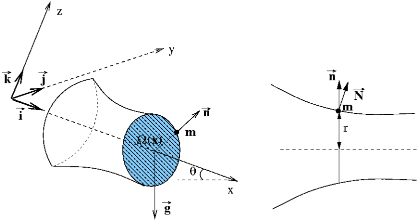
- Euler 3D compressible equations
- Curvilinear transformation
- Asymptotic analysis

are necessary. . .

Toward the PF model

The Euler model in cartesian coordinates

$$
\begin{aligned}
\partial_t \rho + \mathbf{div}(\rho \overrightarrow{U}) &= 0 \\
\partial_t \overrightarrow{U} + \mathbf{div}(\rho \overrightarrow{U} \otimes \overrightarrow{U}) + \nabla p &= \nabla(\overrightarrow{g}.\overrightarrow{OM})
\end{aligned}
$$



After the curvilinear transformation

$$\left\{\begin{array}{rcl} \partial_t(J\rho) + \nabla_{X,Y,Z}\begin{pmatrix}\rho U\\ \rho J V\\ \rho J W\end{pmatrix} & = & 0\\ \partial_t(J\rho U) + \nabla_{X,Y,Z}\left(\rho U\begin{pmatrix}U\\ JV\\ JW\end{pmatrix}\right) + \partial_X p & = & -\rho J g\sin\theta\\ & + & \rho U W \partial_X \theta \end{array}\right.$$

where $(U,V,W)^t = R\overrightarrow{u}$ denotes the reoriented vector, R is the rotation matrix $R = \begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}$,

$J(X,Y,Z) = 1 - Z\partial_X\theta(X)$ using the result

Lemma (Divergence chain rule)

Let $\overrightarrow{\xi} \mapsto \overrightarrow{Y}(\overrightarrow{\xi})$ and $\mathcal{A}^{-1} = \nabla_{\overrightarrow{\xi}} \overrightarrow{Y}$ be the jacobian matrix of the transformation and J its determinant. Then, for any vector field $\overrightarrow{\Phi}$ one has,

$$J \nabla_{\overrightarrow{Y}} . \overrightarrow{\Phi} = \nabla_{\overrightarrow{\xi}} . (J \mathcal{A} \Phi)$$

Classical assumptions : small parameter $\epsilon = H/L$, characteristics dimensions $T, P, \overline{U}, \overline{V}, \overline{W}$, dimensionless quantities $\widetilde{U} = U/\overline{U} \ldots$

Asymptotic analysis

The formal limit when ϵ goes to 0 in physical variables reads,

$$\left\{ \begin{array}{rcl} \partial_t(\rho) + \nabla_{X,Y,Z} \begin{pmatrix} \rho U \\ \rho V \\ \rho W \end{pmatrix} & = & 0 \\ \partial_t(U\rho) + \nabla_{X,Y,Z} \left(\rho U \begin{pmatrix} U \\ V \\ W \end{pmatrix} \right) + \partial_X p & = & -\rho g \sin\theta \\ & - & Z\partial_X(g\cos\theta) \end{array} \right.$$

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Saint-Venant like equations in rigid pipes

We complete the previous system with

- $p = p_a + \dfrac{\rho - \rho_0}{\beta \rho_0}$: linearized pressure law
- $(U, V, W)^t . \overrightarrow{N} = 0$: non penetration condition

Integrating over a cross-section $\Omega(X)$, we get

Conservative variables ($M = \rho A$, $D = M\overline{U}$)

$$\left\{ \begin{array}{rcl} \partial_t(M) + \partial_X(D) & = & 0 \\ \partial_t(D) + \partial_X\left(\dfrac{D^2}{M} + c^2 M\right) + gM\partial_X \widetilde{Z} & = & 0 \end{array} \right.$$

Pseudo-altitude term $\widetilde{Z} = Z + \Phi_\theta - c^2/g \ln(A)$ where

$$\Phi_\theta = \int_{X_0}^{X} R(\xi)\partial_X \cos\theta(\xi)\, d\xi$$

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Conservative variables ($M = \rho A$, $D = M\overline{U}$) with the friction term

$$\left\{\begin{array}{rcl} \partial_t(M) + \partial_X(D) & = & 0 \\ \partial_t(D) + \partial_X\left(\dfrac{D^2}{M} + c^2 M\right) + gM\partial_X \widetilde{Z} & = & -gMK\overline{U}|\overline{U}| \end{array}\right.$$

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Classical properties

Theorem (frictionless)

1. *The system is stricly hyperbolic for* $M > 0$.
2. *For smooth solutions,*
$$\partial_t \overline{U} + \partial_X \left(\frac{\overline{U}^2}{2} + c^2 \ln(M) + g\widetilde{Z} \right) = 0$$
where the steady states for $\overline{U} = 0$*, reads*
$c^2 \ln(M) + g\widetilde{Z} = cte$
3. *It admits a mathematical entropy*
$E(M, D) = \dfrac{D^2}{2M} + Mc^2 \ln M + gM\widetilde{Z}$ *which satisfies the entropy equality*
$$\partial_t E + \partial_X((E + c^2 M)\overline{U}) = 0$$

Classical properties

Theorem (with the friction term)

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$$\partial_t E + \partial_X ((E + c^2 M)\overline{U}) \leqslant -gK\overline{U}^2|\overline{U}|$$

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According to E. Audusse, M-O. Bristeau, B.Perthame . . .

Maxwellian function

Let $\chi : \mathbb{R} \to \mathbb{R}$ be the function such that

$$\chi(w) = \chi(-w) \geq 0, \int_{\mathbb{R}} \chi(w)\, dw = 1, \int_{\mathbb{R}} w^2 \chi(w)\, dw = 1$$

Then we define a Gibbs equilibrium

$$\mathcal{M}(t,x,\xi) = \frac{M}{c}\mathcal{M}\left(\frac{\xi - \overline{U}}{c}\right)$$

which satisfies . . .

$$M = \int_{\mathbb{R}} \mathcal{M}(\xi)\, d\xi,\ D = \int_{\mathbb{R}} \xi \mathcal{M}(\xi) d\xi,\ \frac{D^2}{M} + c^2 M = \int_{\mathbb{R}} \xi^2 \mathcal{M}(\xi)\, d\xi$$

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Macro-microscopic relation

Theorem

(M, D) is an entropic solution of the Saint Venant like system if and only if $\mathcal{M}$ satisfies the kinetic equation,

$$\partial_t \mathcal{M} + \xi . \partial_x \mathcal{M} - g \partial_x \widetilde{Z} . \partial_\xi \mathcal{M} = K(t, x, \xi)$$

where $K(t, x, \xi)$ admits vanishing moments up to order 1 and

$$\int_{\mathbb{R}} \xi^2 K \, d\xi \leq 0, \; a.e.(t, x)$$

How to choose χ ?

Following the idea of Perthame-Simeoni 2001

The only possible choice for χ such that $\mathcal{M}$ satisfies the steady state is $\chi(w) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-w^2}{2}\right)$

Morever,

$$\min\left\{\epsilon(f); f > 0, \int_{\mathbb{R}} f(\xi)\, d\xi = M, \int_{\mathbb{R}} \xi f(\xi)\, d\xi = D\right\}$$

where ϵ is the kinetic convexe functionnal

$$\epsilon(f) = \int_{\mathbb{R}} \frac{\xi^2}{2} f(\xi) + c^2 f(\xi) log(f(\xi)) + c^2 f(\xi) log(c\sqrt{2\pi}) + g\widetilde{Z} f(\xi)\, d\xi$$

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Unfortunately, this function is not compact supported

Nevertheless, we will use the indicator χ function

$$\chi(w) = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(w)$$

which satisfies

- the conservation of the steady state,
- the conservation of the in cell-entropy,

and allows an easy computation of macro-microscopic relation

ABOVE ALL we have a numerical CFL condition!

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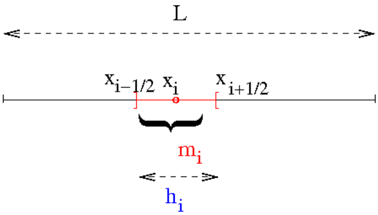
Discretization



- Uniform mesh : $h = \Delta x$
- Discrete macroscopic unknows : M_i^n, D_i^n
- Discrete microscopic unknows : $\mathcal{M}_i^n$
- Discrete macro-microscopic relation :
$$\begin{pmatrix} M_i^n \\ D_i^n \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}_i^n(\xi)\, d\xi$$
- Discrete pseudo-altitude term : $\widetilde{Z} = \widetilde{Z}_i \mathbb{1}_{m_i}(X)$

The kinetic scheme

At time t_n, assuming we know $\mathcal{M}_i^n$
Solving the relaxed problem

$$\begin{cases} \partial_t f + \xi.\partial_X \mathcal{M} - g\partial_X \widetilde{Z} \partial_\xi \mathcal{M} = 0 & (t,X,\xi) \in [t_n, t_{n+1}] \times m_i \times [0,T] \\ f(t_n, X, \xi) = \mathcal{M}(t_n, X, \xi) & (X,\xi) \in m_i \times [0,T] \end{cases}$$

which is discretized as follows

$$\forall i = 0, \ldots, N+1, \; \forall n = 0, \ldots, T,$$

$$f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) - \xi \frac{\Delta_t}{\Delta_X} \left\{ \mathcal{M}_{i+1/2}^-(\xi) - \mathcal{M}_{i-1/2}^+(\xi) \right\}$$

$\Longrightarrow$ we get f_i^{n+1}

Finally, we define

$$\mathcal{U}_i^{n+1} = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f_i^{n+1}(\xi)\, d\xi$$

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...

Remark

The kernel K is not computed : it is a way to perform all collisions at once

It remains to define the interface equilibrium densities

Overpass Reflection

$$\begin{aligned}
\mathcal{M}^{-}_{i+1/2}(\xi) &= \mathbb{1}_{\xi>0}\mathcal{M}^{n}_{i}(\xi) + \mathbb{1}_{\xi<0,\xi^2-2g\Delta\widetilde{Z}_{i+1/2}<0}\mathcal{M}^{n}_{i}(-\xi) \\
&+ \mathbb{1}_{\xi<0,\xi^2-2g\Delta\widetilde{Z}_{i+1/2}>0}\mathcal{M}^{n}_{i+1}\left(-\sqrt{\xi^2-2g\Delta\widetilde{Z}_{i+1/2}}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{+}_{i+1/2}(\xi) &= \mathbb{1}_{\xi<0}\mathcal{M}^{n}_{i+1}(\xi) + \mathbb{1}_{\xi>0,\xi^2+2g\Delta\widetilde{Z}_{i+1/2}<0}\mathcal{M}^{n}_{i+1}(-\xi) \\
&+ \mathbb{1}_{\xi>0,\xi^2+2g\Delta\widetilde{Z}_{i+1/2}>0}\mathcal{M}^{n}_{i}\left(\sqrt{\xi^2+2g\Delta\widetilde{Z}_{i+1/2}}\right)
\end{aligned}$$

Numerical properties of the scheme with the indicator function

Theorem

1. *Assuming the CFL condition* $\max_{i\in\mathbb{Z}}\left(|\overline{U}_i^n| + \sqrt{3}c\right) \leq \frac{\Delta x}{\Delta t}$, *the numerical scheme keeps the pressurized wet area positive* $M_i^n > 0$.
2. *The steady state is preserved* $\overline{U}_i^n = 0, \frac{c^2}{g}\ln(\rho_i^n) + \widetilde{Z}_i = cst$

Remark

The entropy inequality seems to be numerically validate.

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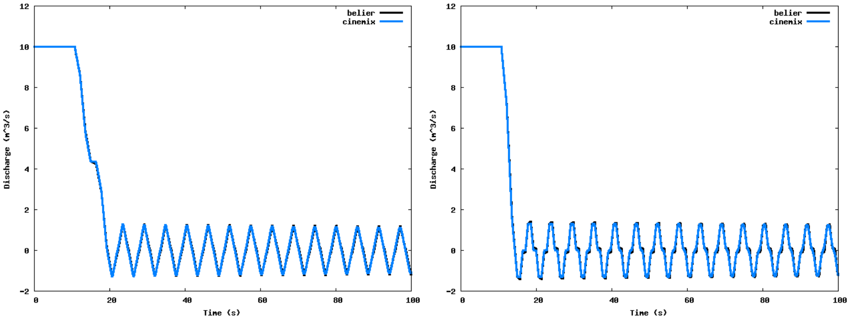
We consider a pipe of length 2000 m, $R = 1\ m$, $CFL = 0.8$, $N = 100$, $\tan\theta = 0,087488664$, $Z0 = 250\ m$ where we consider two type of boundary conditions

1. **upstream** the total head is constant equal to 300 m,
   downstream the discharge is constant to 10 $m^3 s^{-1}$ for $t \leq 10$ and decrease linearly to 0 on $10 \leq t \leq 20$, for $t \geq 20$, the discharge is equal to 0.
2. **upstream** the total head is constant equal to 300 m,
   downstream the discharge is constant to 10 $m^3 s^{-1}$ for $t \leq 10$ and decrease linearly to 0 on $10 \leq t \leq 15$, for $t \geq 15$, the discharge is equal to 0.

The discharge at middle of the pipe



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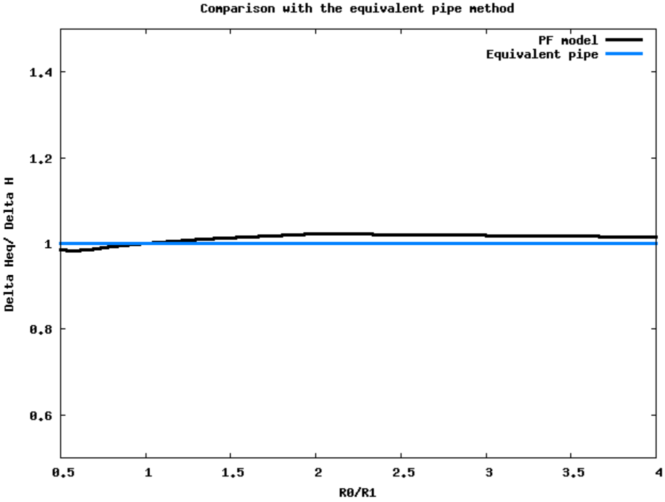
Comparison with the equivalent pipe method

Data and Input for the computation of pressure rise for water hammer at the middle of the pipe

1. $L = 1000\,m$, upstream radius $R_0 = 1\,m$, downstream radius varying $R_1 = 0.25\,m$, $0.5\,m$, $2\,m$.
2. $N = 100$, CFL$= 0.8$
3. The downstream discharge before the shut-down ($3\,s$) is constant equal to $1\,m^3.s^{-1}$.

in the case of contracting-expanding pipe



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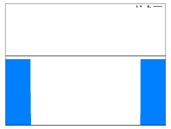
Just for fun

M. hammer



Conclusions

We have a good agreement with the equivalent pipe theory

Future works

- Mixed flows in closed pipes with variable sections
- Air entrapment and cavitation problem