NUMERIWAVES Project ERC FP7 - 246775 New analytical and numerical methods in wave propagation

OPTPDE Summer School *Challenges in Applied Control and Optimal Design*, Bilbao, Basque Country, July 4-8, 2011

basque center for applied mathematics

Principal Investigator: Enrique Zuazua
Members: Corentin Audiard, Mehmet Ersoy, Aurora Marica, Jean-Pierre Puel
http://www.bcamath.org/projects/NUMERIWAVES/
{caudiard,mersoy,marica,jppuel,zuazua}@bcamath.org

IDEAS

Aims & Some recent developments

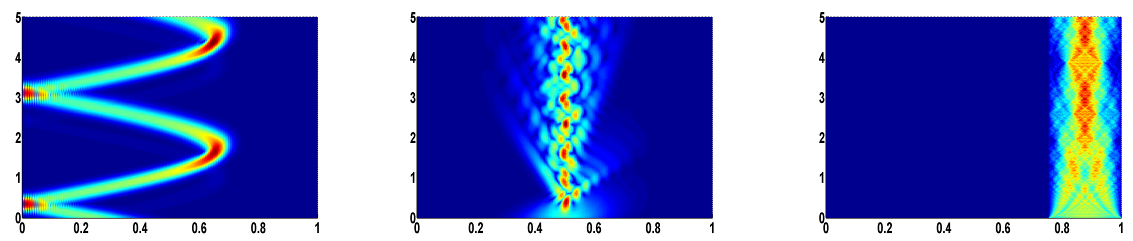
- This project is aimed at *performing a systematic analysis of the combined effect of wave propagation and numerical discretization*, in order to help in the *development of efficient numerical methods mimicking the qualitative properties of continuous waves*. This is an important issue for its many applications: *irrigation channels, flexible multi-structures, aeronautic optimal design, acoustic noise reduction, electromagnetism, water waves, nonlinear optics, nanomechanics, etc.*
- The superposition of the present state of the art in Partial Differential Equations (PDE) and Numerical Analysis is insufficient to understand the spurious high frequency numerical solutions that the interaction of wave propagation and numerical discretization generates. There are some fundamental questions, as, for instance, dispersive properties, unique continuation, control and inverse problems, which are by now well understood in the context of PDE through the celebrated Strichartz and Carleman inequalities, but which are unsolved and badly understood for numerical approximation schemes.
- The aim of this project is to systematically address some of these issues, **developing new analytical and numerical tools**, which require new significant developments, **much beyond the frontiers of classical numerical analysis**, to incorporate ideas and tools from Micro-local and Harmonic Analysis.

This project aims to contribute to develop ***new analytical tools and numerical schemes and will contribute to significant progress in some applied fields in which the issues under consideration play a key role.***

Wave propagation in discrete heterogeneous media

We analyze discrete versions of the so-called observability properties of waves which are relevant in Inverse Problem and Control Theory (cf. [ErZu11]). Observability refers to the possibility to estimate the total energy of the system by continuous measurements on the boundary during a time interval. Fourier analysis can be employed for numerical schemes on uniform grids but not on non-uniform meshes. Combining pseudo-differential calculus and a posteriori error analysis we aim to define notions like the principal symbol of the numerical scheme or its rays of Geometric Optics and to transform the discrete systems into continuous equations with heterogeneous and possible singular coefficients.

Our analysis and numerical simulations exhibit a number of pathological phenomena such as the torsion of the rays of Geometric Optics, fictitious reflection before getting to the boundary. They are illustrated below for the $1-d$ scalar wave equation.



(a) $y=\tan(\pi x/4)$, $y_0=1/4$ (b) symmetric grid w.r.t. $y=1/2$, $y=\sin(\pi x/3)$ for $x\in(0,1/2)$, $y_0=1/2$ (c) uniform grid on $y\in(0,1/4)\cup(3/4,1)$ and $y=1/4+\tan(\pi x/4)$ for $y\in(1/4,3/4)$; $y_0=7/8$

Figure: Propagation of a Gaussian wave packet with initial data $\exp(-\gamma(y-y_0)^2/2)\exp(i\xi_0 y)$ with $h=1/200$; x,y=uniform/non-uniform grid of $(0,1)$ and $\xi_0=\pi/2h_{min}$

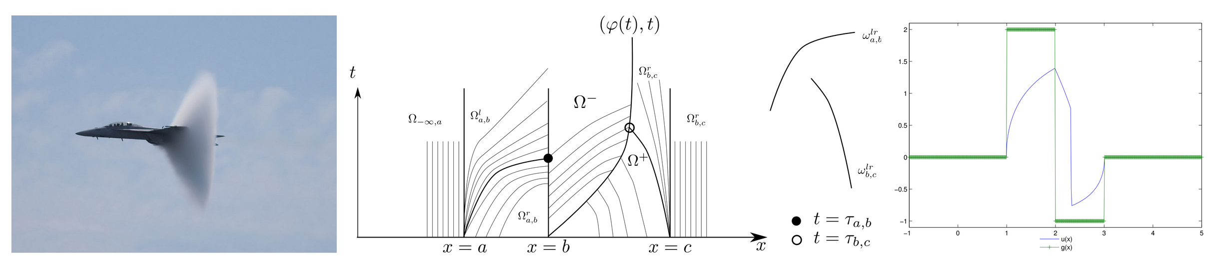
Steady conservation laws in presence of shocks: Applications to control

In [EZ11], we analyze the structure of steady state entropic solutions:

$$\partial_t u+\partial_x f(u)+u=g(x),\quad t>0,\quad u_0(0,x)=u_0(x),\quad x\in\mathbf{R}$$

for $g\in L^1(\mathbf{R})$ with compact support. More precisely we look for time-independent solutions $v=v(x)$. We show, using nonlinear semi-group techniques, that the solutions of the time-evolution problem converge, as $t\to\infty$, towards these steady states. We then characterize the location of its possible shock discontinuities and its sensitivity with respect to small perturbations. These perturbations can be of different nature: the nonlinear flux f, the right hand side force g, the initial data, various parameters entering in the system,…
One of the main motivations of this analysis is to develop new tools to solve control and inverse problems.

A careful analysis of the sensitivity of the shock locations allows to adapt the so-called alternating descent method – introduced in the context of time-dependent scalar conservation laws ([CPZ08]) – to develop more efficient numerical methods than those the standard purely discrete or continuous approaches yield. The methods we develop are faster and potentially they can be extended to multi-d problems involving Navier-Stokes and Euler equations of relevance in aeronautic optimal design ([J82]).



(a) transonic shocks [J82] (b) $(x,t)-$plan for finite time convergence (c) A steady solution

Figure: Transonic shocks & finite time emerging discontinuous steady solution

Qualitative properties of the cKdV equations: dispersive estimates

Our aim is to develop efficient numerical approximation schemes for the solutions of the generalized critical *Korteweg-de Vries* (cKdV) equation

$$\partial_t u+u^4\partial_x u+\partial_x^3 u=0,\quad u|_{t=0}=u_0\in L^2 \qquad \text{(cKdV)}$$

For very rough initial data, the well posedness of the Cauchy problem for small initial data relies on the following dispersive estimates:

$$\left|u\right|_{L_x^5 L_t^{10}}\lesssim\left|u_0\right|_{L^2},\quad \left|\partial_x u\right|_{L_x^\infty L_t^2}\lesssim\left|u_0\right|_{L^2},$$

which *are not satisfied for standard discretization by finite differences*. In general, even the existence of solutions of the semi-discretized schemes on a time interval independent of the discretization is a delicate matter. Thus one has to design *dispersive numerical schemes* that mimick the properties of the continuous solutions.

Our approach, following [IZ09], is to construct an operator Π that filters spurious high frequency numerical components. Our main result (C. Audiard, 2011) is the following:
For $u\in l^2(h\mathbf{Z})$, set $(\partial_h u)_n=(u_{h(n+1)}-u_{hn})/h$,
$(\partial_h^3 u)_n=(u_{h(n+2)}-2u_{h(n+1)}+2u_{h(n-1)}-u_{h(n-2)})/2h^3$.
There exists $N>0$ and a linear interpolation operator

$$\Pi:\ l^2(Nh\mathbf{Z})\to\mathbf{l^2(hZ)},$$

such that the problem

$$\begin{cases}\partial_t u_h+\partial_h\Pi u_h^5/5+\partial_h^3 u_h=0,\\ u_h|_{t=0}=\Pi u_{0h}\in L^2,\end{cases}$$

has an unique solution for $\left|u_0\right|_2$ small enough, satisfying the discrete analogs of the dispersive estimates.
Moreover, if $\Pi u_{0h}\to u_0$ L^2, then u_h converges in L^2_{loc} to the solution of the continuous problem with initial data u_0.

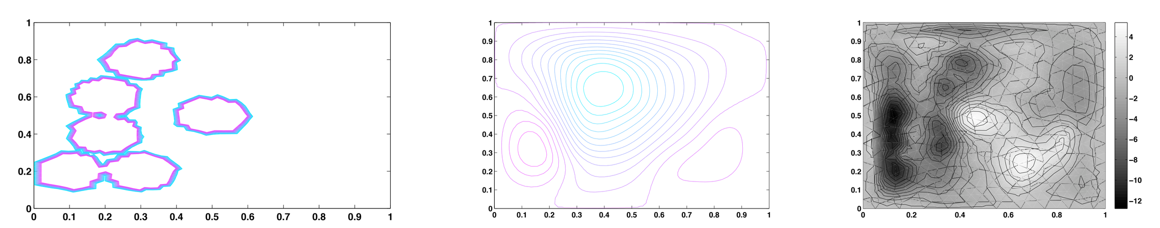
A linear quasigeostrophic large scale ocean model

Our aim is to analyze a data assimilation problem encountered in oceanology to simulate the evolution of the ocean circulation. The model under consideration is

$$\begin{cases}R_o\dfrac{\partial}{\partial t}(\Delta\psi)-\epsilon_m\Delta^2\psi+\epsilon_s\Delta\psi+\dfrac{\partial\psi}{\partial x_1}=-\mathrm{curl}\,\mathcal{T}\quad\text{in }\Omega\times(0,T),\\ \psi=\dfrac{\partial\psi}{\partial n}=0\quad\text{on }\Gamma\times(0,T),\quad\Delta\psi(0)=\Delta\psi_0\quad\text{in }\Omega,\end{cases}$$

where $\psi(t,x)$ is the stream function, R_o, ϵ_s, ϵ_m the *Rossby, Stommel, Munk numbers* resp. and $\mathcal{T}$ the *wind stress.*
Everything is known, EXCEPT the initial value at time $t=0$.
A history of measurements of the solution (observations ψ_{obs}) in some sub-domain $\mathbf{O}$ during the time period $(0,T_0)$ is provided.
A control problem for the adjoint system z is introduced to recover (and reconstruct) the final state value (state value at time $t=T_0$) without knowledge of ψ_0. Let $\varphi_0\in H_0^1(\Omega)$ and h in $L^2(L^2(\mathbf{O}))$ (control).

Then, there exists a control $h(\varphi_0)$ such that $z(0)=0$.
Furthermore, this null controllability problem can be approximated by a standard optimal control problem penalizing, by means of a large parameter α, the constraint $z(0)=0$.
Using this control we can recover the value of the final state $\Delta\psi(T_0)$, by taking successively for φ_0 elements of a Hilbert basis of $L^2(\Omega)$.



(a) Location of the observatories (b) Recovered stream function at T_0 using 6 observatories (c) Relative percentage of error at T_0 with $\alpha=0.025$.

Figure: Results with six observatories

References

[CPZ08] Castro C., Palacios F. and Zuazua E., An alternating descent method for the optimal control of the inviscid Burgers equation in the presence of shocks, Math. Models Methods Appl. Sci., 2008.
[EZ11] Ersoy M. and Zuazua E., Steady conservation laws in presence of shocks: Applications to control, In preparation, 2011.
[ErZu11] Ervedoza S. and Zuazua E., The wave equation: control and numerics, Lecture Notes in Mathematics, CIME Subseries, Springer Verlag, to appear.
[GOP11] Garcia G., Osses A., Puel J.-P., A Null Controllability data assimilation methodology applied to a large scale ocean circulation model, M2AN, 2011.
[IZ09] Ignat L. and Zuazua E., Numerical schemes for the nonlinear Schrödinger equation, SIAM J. Numer. Anal. Volume 47, Issue 2, pp. 1366-1390 (2009).
[J82] Jameson A., Steady-state solution of the Euler equations for transonic flow, Transonic, Shock, and Multidimensional Flows, Advances in Scientific Computing, 1982.
[MZ10] Marica A. and Zuazua E., Localized solutions for the finite difference semi-discretization of the wave equation, C. R. Acad. Sci. Paris, Ser. I 348 (2010) 647ÃŘ652.