Éléments Finis: applications (FreeFem++)

M. Ersoy

Ecole d'ingénieurs de l'Université de Toulon
MOdélisation et CAlculs Fluides et Structures (MOCA)
2A

OUTLINE OF THE TALK

Outline

1. Introduction
2. Examples of linear and non linear PDEs
   - Poisson equation
   - Heat equation
   - Convection problem
   - Advection-diffusion problem
   - Incompressible Navier-Stokes equations
   - many among others
3. Conclusion

A software for solving PDEs

FreeFem++ for 2D-3D [1] PDEs

- Finite element method

1. in progress

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- Export data result to post-treatment with medit, gnuplot, tecplot, ...
- Written and use the same C++ syntaxes

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3. Or with the X-version and click
   `run`
4. Each line end with ;

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Type

- Operators as in C langage : `+ - * ^ == < > <= >= & = +=`
- int, real, complex, string, ...
- `real[int] a(n);`
  `a[3]=2;`
- `real[int,int] a(n,m);`
  `a[1][2]=0; ...`

Functions

- `cos`, `sin`, `tan`, `acos`, `asin`, `atan`, `cosh`, `sinh`, `acosh`, `asinh`, `log`, `log10`, `exp`, `sqrt`

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instruction 1;
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- Simple function :

```
func outvar = expression of x and y and classical functions;
func f = exp(x)*y+sqrt(x)*cos(pi*x);
```

Control instruction

- loop for

```
for (int,cond,incr)
{
...
};
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```
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...
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- control `if`

```
if (cond)
{
...
}
else {
...
}
;
```

Input/Output

- Open to read a file `ifstream`
  ```
  ifstream name(file_name);
  ```

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- ```
  ofstream data("result.dat");
  for (int i=0,i<=100,i++)
  {
  data << " x = " << x[i] << endl
  };
  ```
  ```

How to define the mesh

Let Ω be an arbitrary domain :

- If the domain Ω is regular, for instance rectangle of size $[a, b] \times [c, d]$, then the command
  `mesh mesh_name = square(n,m,[a+(b-a)*x,c+(d-c)*y];`
  generates a regular mesh with triangles of size $n \times m$
  n means that the segment $[a, b]$ is divided into $n + 1$ points.

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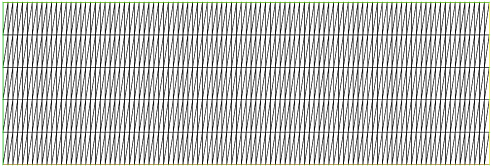
  n means that the segment $[a, b]$ is divided into $n+1$ points.
- Otherwise, the domain Ω has to be defined by a parametrization of its boundary with the following command
  `border border_name(t=beg,end) {x=x(t) ; y = y(t); label = num_label};`
  then, the mesh of Ω is obtained with
  `mesh mesh_name =buildmesh(b1(n1)+b2(n2)+...+bk(nk));` .
  n_i means that the border b_i is divided into $n_i + 1$ points.

Examples

The code see the numerical code generates the mesh



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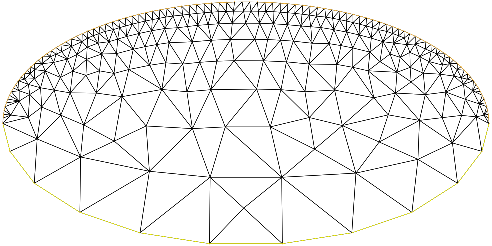

EXAMPLES

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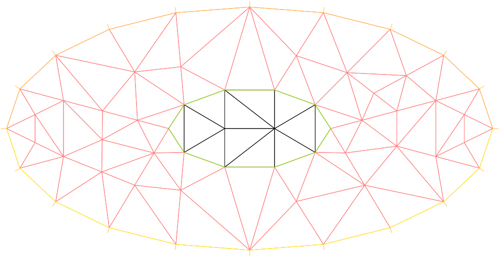

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- readmesh(file_name); read the mesh file file_name
- mesh newmesh = movemesh(oldmesh,[f1(x,y),f2(x,y)]);

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- `mesh newmesh = movemesh(oldmesh,[f1(x,y),f2(x,y)]);`
- `mesh newmesh = adaptmesh(oldmesh,crit);` adapt the mest w.r.t. one or more criterions `crit`

Solve a PDE with Freefem++

- Define the approximation space V_h with the command
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- Define the approximation space V_h with the command
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- Define the variational problem
  `problem problem_name(u,v,solver)=`
  `a(u,v)-l(v)`
  `+ (boundary conditions);` and add the command
  `problem_name;`
  or replace simply `problem` with `solve`.

Bilinear, linear form and boundary conditions

- Bilinear form

  `int2d(dx(u)*dx(v)+dy(u)*dy(v))` $\Longleftrightarrow \displaystyle\int_\Omega \nabla u(x,y) \cdot \nabla v(x,y)\, dx\, dy$

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- Neumann boundary conditions

  `-int1d(mesh_name,border_name)(b*v)` $\iff \int_{\partial\Omega} \nabla u(x,y) \cdot n\; v\; dx\, dy$

  where $\nabla u(x,y) \cdot n = b$

OUTLINE

1. INTRODUCTION

2. EXAMPLES OF LINEAR AND NON LINEAR PDES
   - Poisson equation
   - Heat equation
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3. CONCLUSION

Outline

The problem

Let us consider the following Poisson problem on the unit square

$$\begin{cases} -\Delta\varphi = f & \text{in } \Omega \\ \varphi(x_1, x_2) = 0 & \text{on } \Gamma_2 \cup \Gamma_3 \\ \partial_n \varphi := \nabla\varphi \cdot n = 0 & \text{on } \Gamma_1 \cup \Gamma_4 \end{cases}$$

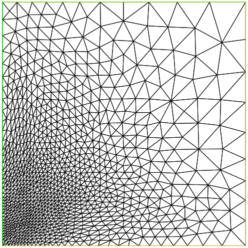

FIGURE: The mesh of Ω. The borders Γ_1 is by default $\{(x,y); y=0\}$, Γ_2 is by default $\{(x,y); x=1\}$, Γ_3 is by default $\{(x,y); y=1\}$, and Γ_4 is by default $\{(x,y); x=0\}$. The label for Γ_i is i.

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Thus, the weak form is for any w test function :

$$\int_\Omega \nabla\varphi \cdot \nabla w \, dx = \int_\Omega f w \, dx + \int_{\Gamma_1 \cup \Gamma_4} 0 w \, ds$$

$$A(\varphi, w) = l(w)$$

with

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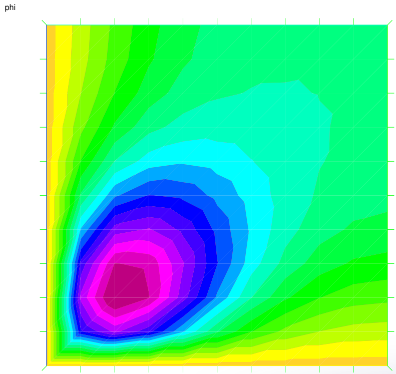

(a) 200 triangles

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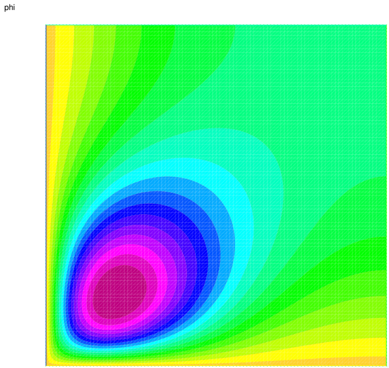

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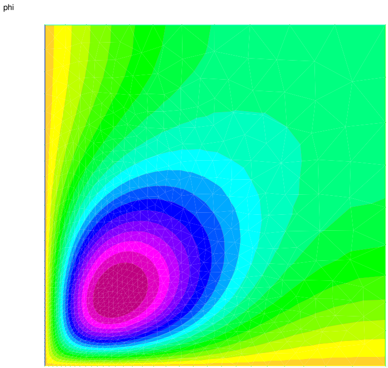

(c) Adaptive mesh (based on $\nabla\phi$) with 1798 triangles (initial mesh 200 triangles)

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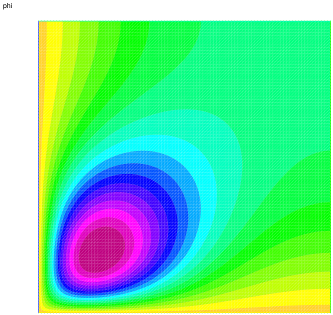
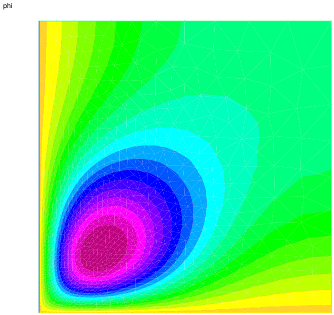


(d) 20000 triangles



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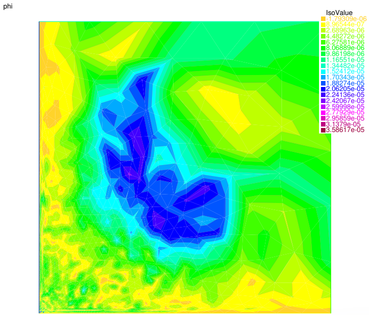

(f) local error : $|\varphi_{20000} - \varphi_{\text{adapt}}|$

The problem

Let us consider the Poisson problem on a ……. . .Poisson domain with homogenous Dirichlet boundary conditions with $f = 1$.

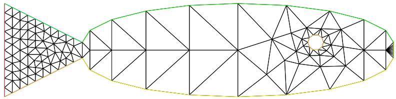

FIGURE: The mesh of Ω with 217 triangles

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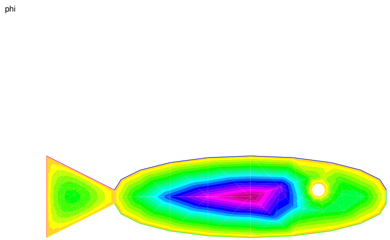

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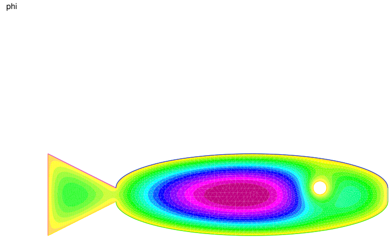

(b) 21125 triangles

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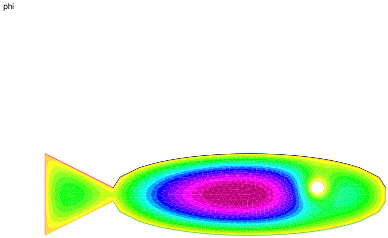

(c) Adaptive mesh (based on $\nabla\phi$) with 5155 triangles (initial mesh 217 triangles)

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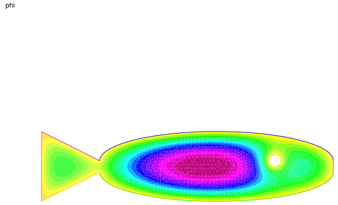

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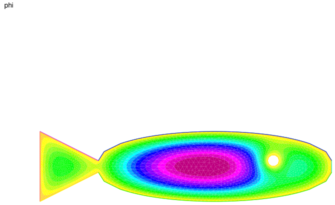

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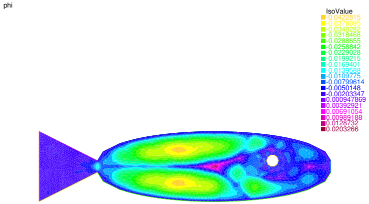

(f) local error

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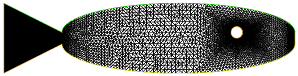
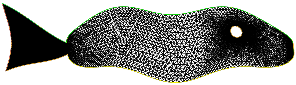
Let us consider the Poisson problem on a ....Poisson domain with homogenous Dirichlet boundary conditions with $f = 1$. One can also apply a deformation to the domain with `movemesh`, here $(x + sin(y\pi)/10, y + cos(x\pi)/10)$:



(g) Initial mesh with 21125 triangles



(h) Deformed mesh with 21125 triangles

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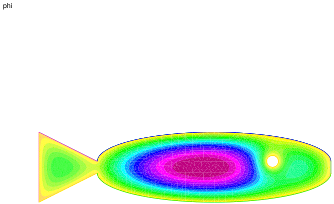
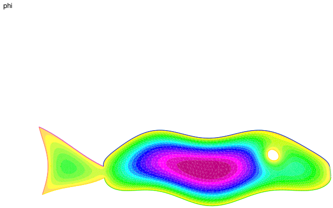


(i) Initial mesh with 21125 triangles



(j) Deformed mesh with 21125 triangles

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  - the command `plot` has several options ...
- It is also possible to export data to use paraview, tecplot, visit or other visualisation software

OUTLINE

The problem

Given f and $\boldsymbol{A}$ p.s.d, let us consider the following heat equation :

$$\begin{cases} \partial_t \varphi - \mathsf{div}(\boldsymbol{A}(t,x)\nabla\varphi) = f & \text{in } t \in (0,T],\ x \in \Omega \\ \varphi(t,x) = z(t,x) & \text{for } t > 0,\ x \in \Gamma_1 \\ \partial_n \varphi := \boldsymbol{A}\nabla\varphi \cdot n = 0 & \text{for } t > 0,\ x \in \Gamma_2 \\ \varphi(0,x) = \varphi_0(x) & \text{for } x \in \Omega \end{cases}$$

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Thus, the weak form of the equation for any "suitable" w test function is :

$$\int_\Omega \partial_t \varphi(t,x) w + \boldsymbol{A}\nabla\varphi(t,x) \cdot \nabla w \, dx = \int_\Omega f(t,x) w \, dx + \text{ BC}$$

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where φ^n is supposed to be an approximation of φ at time $t_n = n\delta t$.

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As a consequence, noting

$$a(t_{n+1}, \varphi, w) = \int_\Omega \frac{\varphi^{n+1}}{\delta t} w + \boldsymbol{A}^{n+1} \nabla \varphi^{n+1} \cdot \nabla w \, dx$$

and

$$l(t_n, w) = \int_\Omega \left(\frac{\varphi^n}{\delta t} + f^n \right) w \, dx + \text{ BC}$$

one has to solve

$$a(t_{n+1}, \varphi, w) = l(t_n, w),\ 0 \leqslant n < N-1.$$

Thus, the Freefem++ code is

```
//Parameters
int N = ...;
real T = ...,dt = ...;
//Define Omega
mesh Th = ...;
//Define FE space and all required functions (especially phi0)
fespace Vh ...;
```

Evolution problems with Freefem++

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for(real t=0;t<=T;t=t+dt)
{
  solve Evolution_Problem(phi,w) =
    ...
  ;
  phi0 = phi;
  plot(...);
}
```

if the stiffness matrix depend on t otherwise

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An exemple

Let us consider the heat equation with $\boldsymbol{A}$ the identity matrix with homogenous Dirichlet boundary conditions and $f = \exp\left(-\sin(t)(x^2+y^2)\right)$ on the Poisson domain.

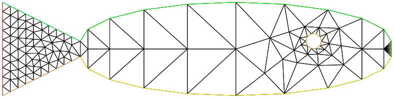

Figure: The mesh of Ω with 217 triangles

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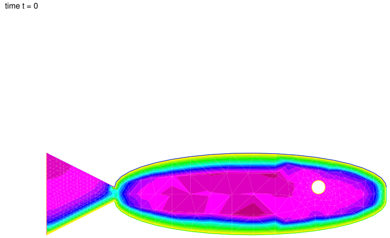
Let us consider the heat equation with $\boldsymbol{A}$ the identity matrix with homogenous Dirichlet boundary conditions and $f = \exp\left(-\sin(t)(x^2+y^2)\right)$ on the Poisson domain.

Then,



(a) t=0.00

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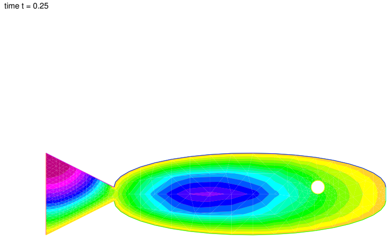

(b) t=0.25

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Let us consider the heat equation with $\boldsymbol{A}$ the identity matrix with homogenous Dirichlet boundary conditions and $f = \exp\left(-\sin(t)(x^2+y^2)\right)$ on the Poisson domain.

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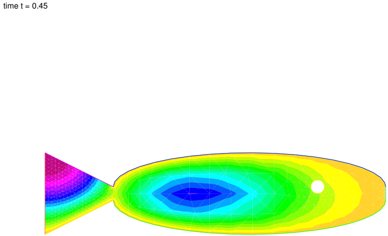

(c) t=0.45

An exemple

Let us consider the heat equation with $\boldsymbol{A}$ the identity matrix with homogenous Dirichlet boundary conditions and $f = \exp\left(-\sin(t)(x^2+y^2)\right)$ on the Poisson domain.

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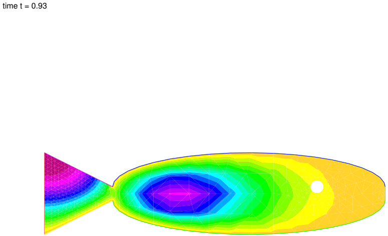

(d) t=0.93

An exemple

Let us consider the heat equation with $\boldsymbol{A}$ the identity matrix with homogenous Dirichlet boundary conditions and $f = \exp\left(-\sin(t)(x^2+y^2)\right)$ on the Poisson domain.

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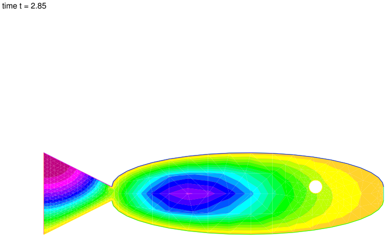

(e) t=2.85

How to visualize ?

As mentionned before one can save the figures. To make a video, one can save at each time step the figure through the command

```
plot(func,cmm="t= "+(t),ps="Folder_Name/File_Name"+num+".eps");
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Here, an example of bash script to do that

```
#!/bin/bash
#Convert eps file to png file
for file in *.eps; do
   convert ./"$file" ./"$file%.eps.png"
done
#Create a movie
mencoder mf:// -mf fps=25:type=png -ovc lavc -oac copy -o
../movie.avi
```

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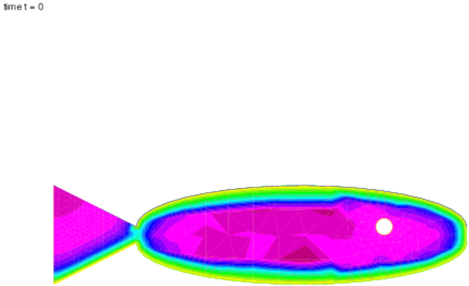

OUTLINE

1. INTRODUCTION

2. EXAMPLES OF LINEAR AND NON LINEAR PDES
   - Poisson equation
   - Heat equation
   - Convection problem
   - Advection-diffusion problem
   - Incompressible Navier-Stokes equations
   - many among others

3. CONCLUSION

A "characteristic Galerkin" method

Let us consider the following convection problem

$$\partial_t u + \boldsymbol{c}(x) \cdot \nabla u = 0, \ (t, x) \in (0, T) \times \Omega$$

with the initial data $u(0, x) = u_0(x), x \in \Omega$ and c assumed to be a regular function.

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Then, the exact solution is $u(t, x) = u_0(X(0; t, x))$ where X solve the ODE

$$X'(s; t, x) = c(X(s)),\ X(t; t, x) = x\ .$$

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We perform this at each t_n. Noting $c = (c_1, c_2)$, the command is simply

```
u = convect([c1,c2],-dt,uold)
```

where convect returns $u \circ X$(t) see the numerical code.

Outline

Applications 1 : advection-diffusion problem

Let f and $c(t,x) \in \mathbb{R}^2$ for all $(t,x) \in [0,T] \times \Omega$ be given functions. Let us consider the following advection-diffusion problem

$$\partial_t u + c \cdot \nabla u - \Delta u = f, x \in \Omega, t > 0$$

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Therefore, one can, for instance, use an implicit Euler scheme in time with the characteristic method :

$$\int_\Omega \frac{u^{n+1} - u \circ X^n}{\delta t} w + \nabla u^{n+1} \cdot \nabla w \; dx = \int_\Omega f w \; dx$$

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Exercice : system of advection-diffusion problem

Let $f(t,x) \in \mathbb{R}^2$ for all $(t,x) \in [0,T] \times \Omega$ be a given function. Let us consider the following coupled advection-diffusion problem of a species i

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Therefore, as done before, using an implicit Euler scheme in time with the characteristic method, we get

$$\int_\Omega \frac{u_1^{n+1} - u_1 \circ X^n}{\delta t} w_1 + \nabla u_1^{n+1} \cdot \nabla w_1 \; dx - \int_\Omega f w_1 \; dx +$$

$$\int_\Omega \frac{u_2^{n+1} - u_2 \circ X^n}{\delta t} w_2 + \nabla u_2^{n+1} \cdot \nabla w_2 \; dx - \int_\Omega f w_2 \; dx = 0$$

with $\dfrac{d}{dt} X = u$.

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with $\frac{d}{dt} X = u$. see the numerical code

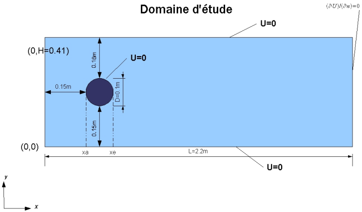
Outline

THE PROBLEM

Let us consider the incompressible Navier-Stokes equation

$$(NSI)\begin{cases} \rho(\partial_t u + (u.\nabla)u) - \rho\nu\Delta u + \nabla p &= 0 \\ \mathrm{div}(u) &= 0 \\ u(x,0) &= u_0(x) \\ + \text{ boundary conditions} & \end{cases}$$

on the domain Ω



with the fluid velocity, $\rho = 1.0$ the density, the viscosity $\nu = 10^{-3}\ m^2/s$ and p the pressure.

VF

Let V be the functional space for u and M the one for p. Let us note the discrete spaces as follows

$$V_h = \{v_h \in V;\, v_{h|K} \in \mathbb{P}_k,\, \forall K \in \tau_h\}$$

and

$$M_h = \{q_h \in M;\, q_{h|K} \in \mathbb{P}_l,\, \forall K \in \tau_h\}$$

where τ_h stands for the mesh and K a given finite element. We fix $k = 2$ and $l = 1$.
Noting $v \in V_h$ and $p \in M_h$ the test functions, one can perform the following implicit scheme

$$\left\{ \begin{array}{rcl} \rho \int_\Omega \frac{u^{n+1}}{\delta t} v\, dx + \rho\nu \int_\Omega \nabla u^{n+1} : \nabla v\, dx - \int_\Omega \mathrm{div}(v) p^{n+1}\, dx & = & \\ \rho \int_\Omega \frac{u^n \circ X^n(x)}{\delta t} v\, dx & & \\ \int_\Omega \mathrm{div}(u^{n+1}) q\, dx & = & 0 \end{array} \right.$$

where the characteristic method is used as in the previous example.

The code is

```
problem pbNSI2D2(u1,u2,p,v1,v2,q,solver=UMFPACK)
  = int2d(Th)( rho/dt*(u1*v1+u2*v2)
             + rho*nu*( dx(u1)*dx(v1)+ dy(u1)*dy(v1)+
                        dx(u2)*dx(v2)+ dy(u2)*dy(v2)
                       )
             -p*dx(v1)-p*dy(v2)
             -q*dx(u1)-q*dy(u2) + perturb*p*q
     )
  -int2d(Th) (rho/dt*convect([u1car,u2car],-dt,u1car)*v1+
              rho/dt*convect([u1car,u2car],-dt,u2car)*v2
      )
  -int1d(Th,3)( g1*v1 +g2*v2 ) // Condition de Neumann
  +on( 3, u2 = 0 )
  +on( 1 , u1 = u0 , u2 = v0 )
  +on( 2 , 4 , 5 , u1 = 0 , u2 = 0 )
  ;
```

see the full numerical code.

Outline

1. Introduction

2. Examples of linear and non linear PDEs
   - Poisson equation
   - Heat equation
   - Convection problem
   - Advection-diffusion problem
   - Incompressible Navier-Stokes equations
   - many among others

3. Conclusion

- The convection-diffusion problem see the numerical code
- The shallow water equations on fixed and moving bottom see the numerical code
- The shallow water and Exner equations see the numerical code
- see the section "Learning by examples" of the freefem++ pdf file.

1. Introduction

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And a

Finally a lot of equations can be quickly solved with freefem++.


Enjoy
yourself