Modélisation d'écoulements des fluides et environnement

Free surface and groundwater flows modeling

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https://l2mias.com/french/ecole-cimpa-2023-chad/

- General context : coastal engineering, sustainable development and climate change
- Application : sandy beaches
  - → 1/3 of beaches are sandy and 1/4 are eroding at rates of 0.5m/year due to rising sea levels [6, 10]
  - → socio-economic impact

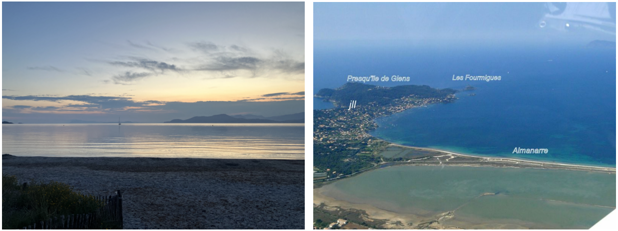

Figure – Almanarre beach, Hyeres, France [a]

a. (source : http ://laurejo.canalblog.com/)

Aims of these lectures :

- Free surface and groundwater modeling
- Dimension reduction techniques
- Numerical method based on Discontinuous Galerkin method
- Applications

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Lectures are organized as follows :

- L1 : Dimension reduction for free surface flows model including recharge
- L2 : Groundwater flows modeling
- L3 : Introduction to the Discontinuous Galerkin (DG) method for transport equation (hyperbolic, c.f. M. Parisot's lectures for the Finite Volume approach)
- L4 : Introduction to the DG method for parabolic-elliptic equations
- L5 : Application of the DG method for a convection-diffusion equation

LECTURE 1 :
Dimension reduction for free surface flows model including recharge

Outline of the Lecture

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Asymptotic Reduction Methods (ARM) provide a powerful way to gain intuition about complex systems without solving them exactly !

- **Simplification of complex mathematical models :**
  - → Many physical systems are governed by complex equations that are difficult to solve or simulate directly.
  - → ARM helps identify dominant balances and approximate the behavior of the system reducing model complexity.

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- **Computational Efficiency :**
  - → Full-scale models are often computationally expensive due to high dimensionality.
  - → ARM yields simplified models that require fewer resources for numerical simulations.

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- Computational Efficiency :
  - → Full-scale models are often computationally expensive due to high dimensionality.
  - → ARM yields simplified models that require fewer resources for numerical simulations.
- **Improved Insight and Analytical Solutions :**
  - → ARM often provides closed-form solutions or simple approximations, making it easier to understand the system's qualitative behavior.
  - → Asymptotic analysis can reveal hidden structures, and stability properties that are hard to identify in the original model.

- Identify Small or Large Parameters :
  - $\rightarrow$ Determine the key parameters ε that are very small or large.
  - $\rightarrow$ Expand the solution in terms of these parameters using asymptotic expansions or perturbation methods.
- Determine the Leading Order Terms :
  - $\rightarrow$ Analyze the eqs to identify which terms dominate as $\varepsilon \rightarrow 0$ or ∞.
  - $\rightarrow$ Neglect higher-order terms that become negligible in the asymptotic reg.
- Simplify the Governing Equations :
  - $\rightarrow$ Derive reduced equations that retain the leading-order behavior and dynamics.
  - $\rightarrow$ This results in simpler ODEs, PDEs, or algebraic equations.
- Validate the Reduced Model :
  - $\rightarrow$ Compare the reduced model's predictions to the full-scale model or experimental data in the asymptotic regime.
  - $\rightarrow$ Ensure that critical behaviors (e.g., stability) are preserved.

- Free surface flow eqs [4, 5, 7]

$$\begin{cases} \mathrm{div}(\rho_0 \mathbf{u}_f) = 0 \\ \partial_t(\rho_0 \mathbf{u}_f) + \mathrm{div}(\rho_0 \mathbf{u}_f \otimes \mathbf{u}_f) - \mathrm{div}(\sigma(\mathbf{u}_f)) - \rho_0 \mathbf{F} = 0 \end{cases} \quad \text{on } \Omega_f$$

where

$$\Omega_f(t) := \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z_b(x, y) < z < \zeta(t, x, y) \right\}$$

- Notations : $(a, b) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T$, and

$$(\mathrm{div}(\mathbf{A}))_i = \sum_{j=1}^{3} \partial_j A_{ij} \text{ for } i = 1, 2, 3$$

- Free surface flow eqs [4, 5, 7]

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$$\Omega_f(t) := \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; z_b(x,y) < z < \zeta(t,x,y)\right\}$$

with

ζ	:	absolute height of the surface ($[L]$)
z_b	:	topography ($[L]$)
$h := \zeta - z_b$	:	water height ($[L]$)
$\mathbf{u}_f = (u_f, v_f, w_f)^T$	:	velocity field ($[L \cdot T^{-1}]$)
$\mathbf{F} = (0, 0, -g)^T$	:	gravity acceleration ($[L \cdot T^{-2}]$)
$\sigma(\mathbf{u}_f) = -p_f \mathbf{I} + 2\mu D(\mathbf{u}_f)$	:	total stress tensor ($[M \cdot L^{-1} \cdot T^{-2}]$)
$D(u) = \frac{1}{2}\left(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T\right)$	:	strain stress tensor
p_f	:	pressure of fluid ($[M \cdot L^{-1} \cdot T^{-2}]$)
ρ_0	:	density ($[M \cdot L^{-3}]$)
$\mu > 0$	:	dynamic viscosity ($[M \cdot L^{-1} \cdot T^{-1}]$)

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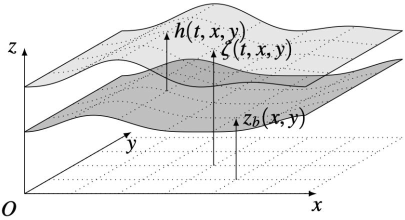
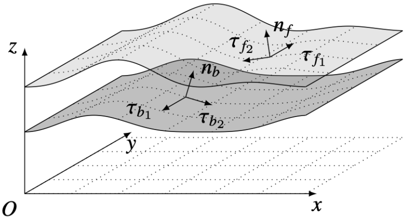


(a) Sketch of variables with h the water height, ζ the free-surface height and z_b the bathymetry



(b) Sketck of basis with $(n_b, \tau_{b_1}, \tau_{b_2})$ on the bathymetry and $(n_f, \tau_{f_1}, \tau_{f_2})$ on the free-surface

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where

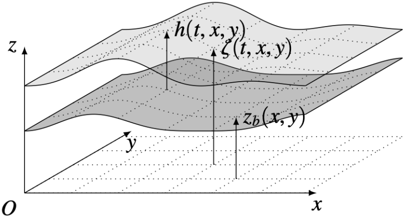
$$\Omega_f(t) := \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; z_b(x, y) < z < \zeta(t, x, y)\right\}$$

- Fluid region indicator function :

$$\Phi(t, x, y, z) := \mathbb{1}_{\Omega_f(t)}(x, y, z) = \mathbb{1}_{z_b(x,y)<z<\zeta(t,x,y)}, \text{ for all } (t, x, y, z) \in \mathbb{R}^4$$

- Φ satisfies the following indicator transport equation :

$$\partial_t \Phi + \partial_x(\Phi u_f) + \partial_y(\Phi v_f) + \partial_z(\Phi w_f) = 0 \text{ on } \Omega_f.$$



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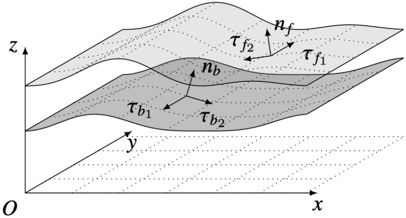


(b) Sketck of basis with $(n_b, \tau_{b_1}, \tau_{b_2})$ on the bathymetry and $(n_f, \tau_{f_1}, \tau_{f_2})$ on the free-surface

- (Navier) Stress boundary condition [4, 7] :

$$(\sigma(\mathbf{u}_f)n_f)\cdot\tau_{f_i} = \mathfrak{M} \text{ on } \mathfrak{F}.$$

where $\mathfrak{M}$ is any meteorological phenomena (such as evaporation, rainfall, wind, etc.), set to 0 in what follows,

$$\mathfrak{F} := \left\{(t,x,y,\zeta) \;\middle|\; t>0,\ (x,y)\in\mathbb{R}^2\right\},$$

the upward normal of $\mathcal{B}$ is defined with :

$$n_f = \frac{1}{\sqrt{1+|\nabla\zeta|^2}}\begin{pmatrix}-\partial_x\zeta\\-\partial_y\zeta\\1\end{pmatrix}$$

and $(\tau_{f_i})_{i=1,2}$ is a basis of the tangential surface :

$$\tau_{f_1} = \frac{1}{|\nabla\zeta|}\begin{pmatrix}-\partial_y\zeta\\\partial_x\zeta\\0\end{pmatrix} \text{ and } \tau_{f_2} = \frac{1}{\sqrt{|\nabla\zeta|^2+|\nabla\zeta|^4}}\begin{pmatrix}-\partial_x\zeta\\-\partial_y\zeta\\-|\nabla\zeta|^2\end{pmatrix}$$

- (Navier) Stress boundary condition [4, 7] :

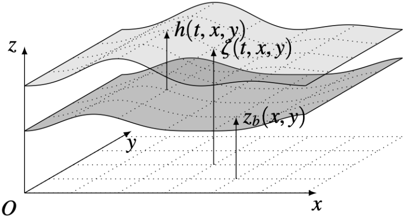
$$(\sigma(\mathbf{u}_f)n_f) \cdot \tau_{f_i} = \mathfrak{M} \text{ on } \mathfrak{F}.$$

- Kinematic boundary condition :

$$\mathbf{u}_f \cdot n_f = \frac{\partial_t \zeta}{\sqrt{1 + |\nabla \zeta|^2}} \text{ on } \mathfrak{F}$$

or using definition of n_f and τ_{f_i} kinematic boundary condition can be rewritten as :

$$\partial_t \zeta + u \partial_x \zeta + v \partial_y \zeta - w = 0.$$



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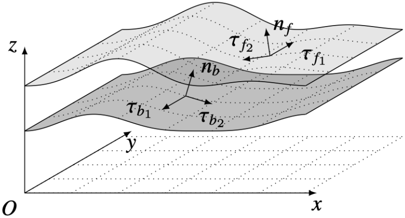

(b) Sketck of basis with $(n_b, \tau_{b_1}, \tau_{b_2})$ on the bathymetry and $(n_f, \tau_{f_1}, \tau_{f_2})$ on the free-surface

- (Navier) Stress boundary condition [3, 4] :

$$(\sigma(\mathbf{u}_f)n_b)\cdot\tau_{b_i} = \left(-\rho_0 k(\mathbf{u}_f)\mathbf{u}_f + \frac{\mu\alpha_{\mathsf{BJ}}}{\sqrt{\mathrm{k}(\psi_g)}}(\mathbf{u}_f - \mathbf{u}_g)\right)\cdot\tau_{b_i} \text{ on } \mathcal{B}$$

where

$$\mathcal{B} := \left\{(x,y,z_b) \;\middle|\; (x,y)\in\mathbb{R}^2\right\},$$

the upward normal of $\mathcal{B}$ is defined with :

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- (Navier) Stress boundary condition [3, 4] :

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and

$C_{lam} \geq 0$	:	laminar friction coefficient
$C_{tur} \geq 0$	:	turbulent friction coefficient
$k(\mathbf{u}_f) = (C_{\mathsf{lam}} + C_{\mathsf{tur}}\lvert\xi\rvert), \forall\xi \in \mathbb{R}^3$	:	a kinematic friction law
$\mathrm{k}(\psi_g) := \mathsf{trace}(\mathbb{k}(\psi_g))$	:	structure of the porous medium
α_{BJ}	:	a dimensionless constant
$\mathbf{u}_g$	:	Darcy velocity field

- (Navier) Stress boundary condition [3, 4] :

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$\mathbf{u}_g$ ($[L\cdot T^{-1}]$) is a function of the hydraulic head h_g ($[L]$) which is a solution of the Richards' equation (RE) (porous media, c.f. Lecture 2, and [1, 2, 8]) :

$$\begin{cases}\mathbf{u}_g = -\mathbb{K}(\psi_g)\nabla h_g \\ \partial_t\theta(\psi_g) + \mathrm{div}(\mathbf{u}_g) = 0\end{cases} \quad \text{in } \Omega_g$$

where the *ground region* (fixed in time) :

$$\Omega_g := \left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; z < z_b(x,y)\right\}$$

where
- θ : water content ([-])
- $\mathbb{K}$: hydraulic conductivity ($[L\cdot T^{-1}]$)
- ψ_g : pressure head ($[L\cdot T^{-1}]$)

- (Navier) Stress boundary condition [3, 4] :

$$(\sigma(\mathbf{u}_f)n_b)\cdot\tau_{b_i} = \left(-\rho_0 k(\mathbf{u}_f)\mathbf{u}_f + \frac{\mu\alpha_{\mathsf{BJ}}}{\sqrt{\mathrm{k}(\psi_g)}}(\mathbf{u}_f - \mathbf{u}_g)\right)\cdot\tau_{b_i} \text{ on } \mathcal{B}$$

- (Coupling) Absorption/Injection condition :

$$\mathbf{u}_f(t,x,y,z)\cdot n_b = \mathbf{u}_g(t,x,y,z)\cdot n_b \text{ on } \mathcal{B}$$

If $\mathbf{u}_g(t,x,y,z)\cdot n_b > 0$, water enters the fluid domain, and if $\mathbf{u}_g(t,x,y,z)\cdot n_b < 0$ water leaves the fluid domain.

- (Navier) Stress boundary condition [3, 4] :

$$(\sigma(\mathbf{u}_f)n_b)\cdot\tau_{b_i} = \left(-\rho_0 k(\mathbf{u}_f)\mathbf{u}_f + \frac{\mu\alpha_{\mathsf{BJ}}}{\sqrt{\mathrm{k}(\psi_g)}}(\mathbf{u}_f - \mathbf{u}_g)\right)\cdot\tau_{b_i} \text{ on } \mathcal{B}$$

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$$\mathbf{u}_f(t,x,y,z)\cdot n_b = \mathbf{u}_g(t,x,y,z)\cdot n_b \text{ on } \mathcal{B}$$

- Pressure condition :

$$-(\sigma(\mathbf{u}_f)n_b)\cdot n_b = \rho_0 g\psi_g \text{ on } \mathcal{B}.$$

Outline

- (Characteristic) Water height H is assumed small with respect to the horizontal length L of the domain and vertical variations W_f are small compared to the horizontal U_f ones :

$$\varepsilon := \frac{H}{L} = \frac{W_f}{U_f} \ll 1$$

$\rightarrow$ Fluid and ground characteristic time : $T_f = \dfrac{L}{U_f} = \dfrac{H}{W_f} = \varepsilon^{\delta} T_g$ with $T_g = \dfrac{L}{U_g} = \dfrac{H}{W_g}$ where $\delta \in \mathbb{R}_+^*$, a parameter that allows us to control the difference between speeds in the fluid and ground domains. As a consequence, one has

$$U_g = \varepsilon^{\delta} U_f \text{ and } W_g = \varepsilon\, \varepsilon^{\delta} U_f.$$

- The pressure scale is defined as :

$$P_f := \rho_0 U_f^2.$$

- Introduce the dimensionless quantities of time $\tilde{t}_f$, space $(\tilde{x}, \tilde{y}, \tilde{z})$, pressure $\tilde{p}_f$, and velocity field $(\tilde{u}_f, \tilde{v}_f, \tilde{w}_f)$ via the following scaling relations :

$$\begin{cases} \tilde{t}_f := \dfrac{t}{T_f}, & \tilde{p}_f := \dfrac{p_f}{P_f} & \tilde{\mathbb{K}} := \mathcal{K}^{-1}\mathbb{K} & \tilde{u}_g := \dfrac{u_g}{U_g} \\ \tilde{x} := \dfrac{x}{L},\ \tilde{y} := \dfrac{y}{L}, & \tilde{u}_f := \dfrac{u_f}{U_f},\ \tilde{v}_f := \dfrac{v_f}{U_f} & \tilde{h}_g := \dfrac{h_g}{H} & \tilde{v}_g := \dfrac{v_g}{V_g} \\ \tilde{z} := \dfrac{z}{H} = \dfrac{z}{\varepsilon L}, & \tilde{w}_f := \dfrac{w_f}{V_f} = \dfrac{w_f}{\varepsilon U_f} & \tilde{\psi}_g := \dfrac{\psi_g}{H} & \tilde{w}_g := \dfrac{w_g}{W_g} \end{cases}$$

with

$$\mathcal{K} = \varepsilon^{\delta} U_f \begin{pmatrix} \dfrac{1}{\varepsilon} & 0 & 0 \\ 0 & \dfrac{1}{\varepsilon} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

- The laminar and turbulent friction factors are scaled, respectively,

$$C_{\mathsf{lam},0} := \frac{C_{\mathsf{lam}}}{V_f} = \frac{C_{\mathsf{lam}}}{\varepsilon U_f}, \quad C_{\mathsf{tur},0} := \frac{C_{\mathsf{tur}}}{\varepsilon}.$$

- The dimensionless number α_{BJ} is rescaled as :

$$\alpha_{\mathsf{BJ},0} := \frac{\alpha_{\mathsf{BJ}}}{\gamma} \text{ with } \gamma = \varepsilon^{\frac{\delta+1}{2}}.$$

- Finally, the following non-dimensional numbers are defined as :

Froude's number,	Fr	$:= U_f/\sqrt{gH}$,
Reynolds number with respect to μ,	Re	$:= \rho_0 U_f L/\mu$.

- Dimensionless incompressible Navier-Stokes equations :

$$\begin{cases} \mathrm{div}_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f) + \partial_{\tilde{z}}\tilde{w}_f = 0 \\ \partial_{\tilde{t}}\tilde{\mathbf{u}}_f + \mathrm{div}_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f \otimes \tilde{\mathbf{u}}_f) + \partial_{\tilde{z}}(\tilde{w}_f\tilde{\mathbf{u}}_f) + \nabla_{\tilde{x}\tilde{y}}\tilde{p}_f = \\ \qquad \mathrm{Re}^{-1}\left(2\mathrm{div}_{\tilde{x}\tilde{y}}(D_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f)) + \nabla_{\tilde{x}\tilde{y}}(\partial_{\tilde{z}}\tilde{w}_f) + \frac{1}{\varepsilon}\partial_{\tilde{z}\tilde{z}}\tilde{\mathbf{u}}_f\right) \\ \partial_{\tilde{z}}\tilde{p}_f = \mathrm{Re}^{-1}\left(\varepsilon^2\delta xtyt\tilde{w}_f + \mathrm{div}_{\tilde{x}\tilde{y}}(\partial_{\tilde{z}}\tilde{\mathbf{u}}_f) + 2\partial_{\tilde{z}\tilde{z}}\tilde{w}_f\right) \\ \qquad -\varepsilon^2\left(\partial_{\tilde{t}}\tilde{w}_f + \mathrm{div}_{\tilde{x}\tilde{y}}(\tilde{w}_f\tilde{\mathbf{u}}_f) + \partial_{\tilde{z}}(\tilde{w}_f^2)\right) - \mathrm{Fr}^{-2} \end{cases}$$

- Dimensionless Richards' equation :

$$\partial_{\tilde{t}}\theta(H\tilde{\psi}_g) + \partial_{\tilde{x}}\tilde{u}_g + \partial_{\tilde{y}}\tilde{v}_g + \partial_{\tilde{z}}\tilde{w}_g = 0$$

- Navier boundary condition on $\mathcal{B}$ with $X = \tilde{u}_f$ and $\tilde{v}_f$:

$$\frac{\partial_{\tilde{z}} X_f}{\varepsilon^2 \mathsf{Re}} = -\left(C_{\mathsf{lam},0} + C_{\mathsf{tur},0}\sqrt{\tilde{u}_f^2 + \tilde{v}_f^2}\right) X_f + \frac{1}{\sqrt{\varepsilon}\sqrt{\mathsf{Re}}\mathsf{Fr}} \frac{\alpha_{\mathsf{BJ},0}}{\sqrt{\tilde{K}_x + \tilde{K}_y}} (X_f - \varepsilon^{\delta} X_g) + O(\mathsf{Re}^{-1}) + O(\varepsilon^2).$$

- Permeable boundary condition $\mathcal{B}$:

$$-\tilde{u}_f \partial_{\tilde{x}} \tilde{z}_b - \tilde{v}_f \partial_{\tilde{y}} \tilde{z}_b + \tilde{w}_f = -\varepsilon^{\delta} \tilde{u}_g \partial_{\tilde{x}} \tilde{z}_b - \varepsilon^{\delta} \tilde{v}_g \partial_{\tilde{y}} \tilde{z}_b + \varepsilon^{\delta} \tilde{w}_g$$

- Balance of pressure on $\mathcal{B}$:

$$\tilde{p}_f = \frac{1}{\mathsf{Fr}^2}\tilde{\psi}_g - \mathsf{Re}^{-1}(2\partial_{\tilde{x}}\tilde{z}_b\partial_{\tilde{z}}\tilde{u}_f + 2\partial_{\tilde{y}}\tilde{z}_b\partial_{\tilde{z}}\tilde{v}_f - 2\partial_{\tilde{z}}\tilde{w}_f) + O(\varepsilon^2)$$

- Kinematic boundary condition on $\mathfrak{F}$: $\partial_{\tilde{t}}\tilde{\zeta} + \tilde{u}_f \partial_{\tilde{x}}\tilde{\zeta} + \tilde{v}_f \partial_{\tilde{y}}\tilde{\zeta} - \tilde{w}_f = 0$
- Stress boundary condition on $\mathfrak{F}$: $\dfrac{\partial_{\tilde{z}}\tilde{u}_f}{\varepsilon^2 \mathsf{Re}} = O(\mathsf{Re}^{-1})$ and $\dfrac{\partial_{\tilde{z}}\tilde{v}_f}{\varepsilon^2 \mathsf{Re}} = O(\mathsf{Re}^{-1})$

- Drop all the term of $O(\varepsilon)$: hydrostatic approximation of the dimensionless NS eqs :

$$
\begin{aligned}
\mathrm{div}_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f) + \partial_{\tilde{z}}\tilde{w}_f &= 0 \\
\partial_{\tilde{t}}\tilde{\mathbf{u}}_f + \mathrm{div}_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f \otimes \tilde{\mathbf{u}}_f) + \partial_{\tilde{z}}\left[\tilde{w}_f\tilde{\mathbf{u}}_f\right] + \nabla_{\tilde{x}\tilde{y}}\tilde{p}_f &= \mathrm{Re}^{-1}\Big(2\mathrm{div}_{\tilde{x}\tilde{y}}(D_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f)) \\
&\quad + \frac{1}{\varepsilon^2}\partial_{\tilde{z}\tilde{z}}\tilde{\mathbf{u}}_f + \partial_{\tilde{z}}\left[\nabla_{\tilde{x}\tilde{y}}(\tilde{w}_f)\right]\Big) \\
\partial_{\tilde{z}}\tilde{p}_f &= \mathrm{Re}^{-1}\Big(\partial_{\tilde{z}}\left[\mathrm{div}_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f)\right] \\
&\quad + 2\partial_{\tilde{z}\tilde{z}}\tilde{w}_f\Big) - \mathrm{Fr}^{-2}
\end{aligned}
$$

- Turbulent (asymptotic) regime consideration : $\mathrm{Re}^{-1} = \varepsilon$ and drop all the term of $O(\varepsilon)$:

$$
\begin{aligned}
\mathrm{div}_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f) + \partial_{\tilde{z}}\tilde{w}_f &= 0 \\
\partial_{\tilde{t}}\tilde{\mathbf{u}}_f + \mathrm{div}_{\tilde{x}\tilde{y}}(\tilde{\mathbf{u}}_f \otimes \tilde{\mathbf{u}}_f) + \partial_{\tilde{z}}\left[\tilde{w}_f\tilde{\mathbf{u}}_f\right] + \nabla_{\tilde{x}\tilde{y}}\tilde{p}_f &= \partial_{\tilde{z}}\left[\frac{1}{\varepsilon}\partial_{\tilde{z}}\tilde{\mathbf{u}}_f\right] \\
\partial_{\tilde{z}}\tilde{p}_f &= -\mathrm{Fr}^{-2}
\end{aligned}
$$

By dropping $\tilde{\cdot}$ we obtain

- Free surface first order approximation :

$$\begin{aligned}
\partial_x u_{f,\varepsilon} + \partial_y u_{f,\varepsilon} + \partial_z w_{f,\varepsilon} &= 0, \\
\partial_t u_{f,\varepsilon} + \partial_x \left[u_{f,\varepsilon}^2\right] + \partial_y \left[u_{f,\varepsilon} v_{f,\varepsilon}\right] + \partial_z \left[u_{f,\varepsilon} w_{f,\varepsilon}\right] + \partial_x p_{f,\varepsilon} &= \partial_z \left[\frac{1}{\varepsilon} \partial_z u_{f,\varepsilon}\right], \\
\partial_t v_{f,\varepsilon} + \partial_x \left[u_{f,\varepsilon} v_{f,\varepsilon}\right] + \partial_y \left[v_{f,\varepsilon}^2\right] + \partial_z \left[v_{f,\varepsilon} w_{f,\varepsilon}\right] + \partial_y p_{f,\varepsilon} &= \partial_z \left[\frac{1}{\varepsilon} \partial_z v_{f,\varepsilon}\right], \\
\partial_z p_{f,\varepsilon} &= -\mathsf{Fr}^{-2}
\end{aligned}$$

  with $(u_{f,\varepsilon}, v_{f,\varepsilon}, w_{f,\varepsilon}, p_{f,\varepsilon})$ the solution of the first-order dimensionless Navier-Stokes system.

- Ground first order approximation :

$$\partial_t \theta(H\psi_g) + \partial_{\tilde{x}} u_g + \partial_y v_g + \partial_z w_g = 0$$

  with $(u_{g,\varepsilon}, v_{g,\varepsilon}, w_{g,\varepsilon}, \psi_{g,\varepsilon}, h_{g,\varepsilon})$ the solution of the first-order dimensionless Richards'equation.

- On $\mathcal{B}$:

$$\frac{1}{\varepsilon}\partial_z \mathbf{u}_{f,\varepsilon} = -k_0(\mathbf{u}_{f,\varepsilon})\mathbf{u}_{f,\varepsilon} + \frac{\mathsf{Fr}^{-1}\alpha_{\mathsf{BJ},0}}{\sqrt{K_x + K_y}}(\mathbf{u}_{f,\varepsilon} - \varepsilon^{\delta}\mathbf{u}_{g,\varepsilon})$$

$$u_{f,\varepsilon}\partial_x z_b v_{f,\varepsilon}\partial_y z_b - w_{f,\varepsilon} = \varepsilon^{\delta} u_{g,\varepsilon}\partial_x z_b + \varepsilon^{\delta} v_{g,\varepsilon}\partial_y z_b - \varepsilon^{\delta} w_{g,\varepsilon}$$

$$p_{f,\varepsilon} = \frac{1}{\mathsf{Fr}^2}\psi_{g,\varepsilon}$$

with $k_0(\mathbf{u}_{f,\varepsilon}) := C_{\mathsf{lam},0} + C_{\mathsf{tur},0}|\mathbf{u}_{f,\varepsilon}|$

- On $\mathfrak{F}$:

$$\frac{1}{\varepsilon}\partial_z \mathbf{u}_{f,\varepsilon} = 0$$

$$\partial_t \zeta + u_{f,\varepsilon}\partial_x \zeta + v_{f,\varepsilon}\partial_y \zeta - w_{f,\varepsilon} = 0$$

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Vertically integrating $\partial_z p_{f,\varepsilon} = -\mathsf{Fr}^{-2}$ between z and $\zeta(t,x,y)$, the hydrostatic pressure is obtained

$$\int_z^\zeta \partial_z p_{f,\varepsilon} dz = -\int_z^\zeta \mathsf{Fr}^{-2} dz$$
$$p_{f,\varepsilon}(t,x,y,\zeta) - p_{f,\varepsilon}(t,x,y,z) = -\mathsf{Fr}^{-2}(\zeta(t,x,y) - z)$$

Assuming that the pressure exerted on the free-surface $p_{f,\varepsilon}(t,x,y,\zeta) = p_{\mathsf{atm}}$ for some constant $p_{\mathsf{atm}} \in \mathbb{R}$ (all other meteorological phenomena are neglected), this becomes

$$p_{f,\varepsilon}(t,x,y,z) = \mathsf{Fr}^{-2}(\zeta(t,x,y) - z) + p_{\mathsf{atm}}$$

- Integrating Indicator transport equation between $z = z_b(x, y)$ and $z = \zeta(t, x, y)$:

$$\int_{z_b}^{\zeta} \partial_t \Phi dz + \int_{z_b}^{\zeta} \partial_x (\Phi u_{f,\varepsilon}) dz + \int_{z_b}^{\zeta} \partial_y (\Phi v_{f,\varepsilon}) dz + \int_{z_b}^{\zeta} \partial_z (\Phi w) dz = 0$$

$$\iff$$

$$\partial_t h(t, x, y) + \partial_x \left(\int_{z_b}^{\zeta} u_{f,\varepsilon} dz \right) + \partial_y \left(\int_{z_b}^{\zeta} v_{f,\varepsilon} dz \right)$$
$$+ \left(u_{f,\varepsilon} \partial_x z_b + v_{f,\varepsilon} \partial_y z_b - w \right) |_{z=z_b}$$
$$- \left(\partial_t \zeta + u_{f,\varepsilon} \partial_x \zeta + v_{f,\varepsilon} \partial_y \zeta - w \right) |_{z=\zeta} = 0$$

- Using the permeable boundary condition and the kinematic one :

$$\partial_t h(t, x, y) + \partial_x \left(\int_{z_b}^{\zeta} u dz \right) + \partial_y \left(\int_{z_b}^{\zeta} v dz \right) = -\varepsilon^{\delta} u_{g,\varepsilon} \partial_x z_b - \varepsilon^{\delta} v_{g,\varepsilon} \partial_y z_b$$
$$+ \varepsilon^{\delta} w_{g,\varepsilon}.$$

- Noting $\bar{f}$ as the mean of a generic function f over the section $[z_b(x,y), \zeta(t,x,y)]$,

$$\bar{f}(t,x,y) = \frac{1}{h(t,x,y)} \int_{z_b(x,y)}^{\zeta(t,x,y)} f(t,x,y,\eta) d\eta,$$

- Using the following approximations :

$$u_{f,\varepsilon}(t,x,y,z) = \bar{u}_\varepsilon + O(\varepsilon) \text{ and } \overline{u_{f,\varepsilon}^2} = \bar{u}_\varepsilon^2 + O(\varepsilon),$$

and dropping the first higher order terms in ε gives a mass-balance equation :

$$\partial_t [h] + \partial_x [h\bar{u}_\varepsilon] + \partial_y [h\bar{v}_\varepsilon] = -\varepsilon^\delta u_{g,\varepsilon}\partial_x z_b - \varepsilon^\delta v_{g,\varepsilon}\partial_y z_b + \varepsilon^\delta w_{g,\varepsilon}.$$

Similarly, we get

$$\partial_t \left[h\bar{u}_\varepsilon\right] + \partial_x \left[h\bar{u}_\varepsilon^2 + \frac{h^2}{2\mathsf{Fr}^2}\right] + \partial_y \left[h\bar{u}_\varepsilon\bar{v}_\varepsilon\right] = -\frac{1}{\mathsf{Fr}^2} h\partial_x \left[z_b\right]$$
$$- k_0(\mathbf{u}_{f,\varepsilon})u_{f,\varepsilon} + \frac{\mathsf{Fr}^{-1}\alpha_{\mathsf{BJ},0}}{\sqrt{K_x + K_y}}(u_{f,\varepsilon} - \varepsilon^\delta u_{g,\varepsilon})$$
$$+ (-\varepsilon^\delta u_{g,\varepsilon}\partial_x z_b - \varepsilon^\delta v_{g,\varepsilon}\partial_y z_b + \varepsilon^\delta w_{g,\varepsilon})u_{f,\varepsilon}$$

and

$$\partial_t \left[h\bar{v}_\varepsilon\right] + \partial_x \left[h\bar{u}_\varepsilon\bar{v}_\varepsilon\right] + \partial_y \left[h\bar{v}_\varepsilon^2 + \frac{h^2}{2\mathsf{Fr}^2}\right] = -\frac{1}{\mathsf{Fr}^2} h\partial_y \left[z_b\right]$$
$$- k_0(\mathbf{u}_{f,\varepsilon})v_{f,\varepsilon} + \frac{\mathsf{Fr}^{-1}\alpha_{\mathsf{BJ},0}}{\sqrt{K_x + K_y}}(v_{f,\varepsilon} - \varepsilon^\delta v_{g,\varepsilon})$$
$$+ (-\varepsilon^\delta u_{g,\varepsilon}\partial_x z_b - \varepsilon^\delta v_{g,\varepsilon}\partial_y z_b + \varepsilon^\delta w_{g,\varepsilon})v_{f,\varepsilon}$$

Dropping $\bar{\cdot}$, we get

$$
\begin{cases}
\partial_t [h] + \partial_x [hu_f] + \partial_y [hv_f] \\
\qquad = \varepsilon^\delta(-u_g\partial_x z_b - v_g\partial_y z_b + w_g) \\
\partial_t [hu_f] + \partial_x \left[hu_f^2 + \dfrac{h^2}{2\mathsf{Fr}^2}\right] + \partial_y [hu_f v_f] = -\dfrac{1}{\mathsf{Fr}^2} h\partial_x [z_b] \\
\qquad - k_0(u_f, v_f)u_f + \dfrac{\mathsf{Fr}^{-1}\alpha_{\mathsf{BJ},0}}{\sqrt{K_x + K_y}}(u_f - \varepsilon^\delta u_g) \\
\qquad + \varepsilon^\delta(-u_g\partial_x z_b - v_g\partial_y z_b + w_g)u_f \\
\partial_t [hv_f] + \partial_x [hu_f v_f] + \partial_y \left[hv_f^2 + \dfrac{h^2}{2\mathsf{Fr}^2}\right] = -\dfrac{1}{\mathsf{Fr}^2} h\partial_y [z_b] \\
\qquad - k_0(u_f, v_f)v_f + \dfrac{\mathsf{Fr}^{-1}\alpha_{\mathsf{BJ},0}}{\sqrt{K_x + K_y}}(v_f - \varepsilon^\delta v_g) \\
\qquad + \varepsilon^\delta(-u_g\partial_x z_b - v_g\partial_y z_b + w_g)v_f
\end{cases}
$$

Validation and discussion on the parameter δ and two-coupling justification

- ε^{δ} : influence of ground flow on free-surface flow valid only for specific values of δ.
- For establishing the classic Shallow-Water system, terms of order greater than ε are dropped : range of validity for δ is $0 \leq \delta < 1$.
- $\delta \in]0, 1[$: two-way coupling is valid if $U_g \approx U_f \iff T_g \approx T_f$, i.e. $\delta \lesssim 1$ specific to the permeability of the ground (coarse grained beaches).
- Hydraulic conductivity in horizontal directions is greater than in vertical directions. This characteristic is observed and documented in the literature [9, pp. 100-103]. He states that K_x/K_z, with K_x and K_z respectively horizontal and vertical hydraulic conductivity, usually fall in the range 2 to 10 for alluvium, but values up to 100 or mode occur where clay layers are present.

Consider that $0 \leq \delta < 1$ and multiply eqs by $\dfrac{HU^2}{L}$ gives the Saint-Venant system with ground influence in its dimensional form :

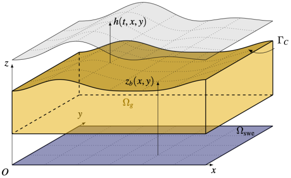
$$\begin{cases} \partial_t [h] + \partial_x [hu_f] + \partial_y [hv_f] = I \\ \partial_t [hu_f] + \partial_x \left[hu_f^2 + g\dfrac{h^2}{2}\right] + \partial_y [hu_f v_f] \\ \qquad = -k(u_f, v_f)u_f + \dfrac{\alpha_{\mathsf{BJ}}}{\sqrt{k_x + k_y}}(u_f - u_g) + Iu_f - gh\partial_x [z_b] \\ \partial_t [hv_f] + \partial_x [hu_f v_f] + \partial_y \left[hv_f^2 + g\dfrac{h^2}{2}\right] \\ \qquad = -k(u_f, v_f)v_f + \dfrac{\alpha_{\mathsf{BJ}}}{\sqrt{k_x + k_y}}(v_f - v_g) + Iv_f - gh\partial_y [z_b] \end{cases}$$

with $I = -u_g \partial_x z_b - v_g \partial_y z_b + w_g$ the quantity of water that enters (I>0) or leaves (I<0) the fluid domain.

Finally, the two-way coupled model of SWE and RE is :

$$
\begin{cases}
\partial_t h + \mathrm{div}(\mathbf{q}) = I, & \text{in } \Omega_{\mathsf{swe}}, \\
\partial_t \mathbf{q} + \mathrm{div}\left(\dfrac{\mathbf{q}\otimes\mathbf{q}}{h} + g\dfrac{h^2}{2}\mathbb{I}\right) = -k(\mathbf{u}_f)\mathbf{u}_f + \dfrac{\alpha_{\mathsf{BJ}}}{\sqrt{k_x + k_y}}(\mathbf{u}_f - \mathbf{u}_g) + I\mathbf{u}_f & \\
\qquad - gh\nabla z_b, & \text{in } \Omega_{\mathsf{swe}}, \\
I = \mathbf{u}_g \cdot (-\partial_x z_b, -\partial_y z_b, 1)^T, & \text{in } \Omega_{\mathsf{swe}}, \\
\mathbf{u}_g = -\mathbb{K}(\psi_g)\nabla h_g, & \text{in } \Omega_g, \\
\partial_t \theta(\psi_g) + \mathrm{div}(\mathbf{u}_g) = 0, & \text{in } \Omega_g, \\
h_g = h + z_b, & \text{on } \Gamma_C, \\
h_g = h_D, & \text{on } \Gamma_D, \\
-\mathbf{u}_g \cdot \mathbf{n} = q_N, & \text{on } \Gamma_N.
\end{cases}
$$

with $\mathbf{q} = \mathbf{u}_f h$, $\Omega_g \subset \mathbb{R}^d \Rightarrow \Omega_{\mathsf{swe}} \subset \mathbb{R}^{d-1}$ with $d = 2, 3$.



By Asymptotic Reduction Methods, we have

- identified small parameter ε and specific asymptotic regime
- expanded the solution in terms of ε
- analyzed the eqs and identified dominant terms as $\varepsilon \to 0$
- retained leading-order terms to derive the hydrostatic approximation
- vertically averaged these eqs to get the Saint-Venant system with recharge
- this results in a simpler form (loss of one dimension)
- formally justified the two-way coupling

To do,

- Derivation of the Richards' equation for saturated/unsaturated porous media (c.f. Lecture 2)
- Introduction to the Discontinuous Galerkin (DG) method for transport equation (hyperbolic, c.f. M. Parisot's lectures for the Finite Volume approach)
- Introduction to the DG method for parabolic-elliptic equations
- Application of the DG method for a convection diffusion equation

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