Modélisation d'écoulements des fluides et environnement

Free surface and groundwater flows modeling

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Last lecture and aims

We have presented

- the (formal) derivation of the SWE and boundary conditions in the framework of Free surface and groundwater flow modeling
- the (formal) derivation of the Richards' equation by upscaling approaches
- the hydraulic properties (or closure)

To do,

- Introduction to the Discontinuous Galerkin (DG) method for transport equation (hyperbolic, c.f. M. Parisot's lectures for the Finite Volume approach)

Numerical methods [1, 8]

Class of numerical methods

- Finite Difference Method (FDM)
  - → Approximates derivatives using differences between adjacent grid points.
  - → Simple implementation and useful for regular geometries.
  - → Common for time-dependent problems like the heat equation.
- Finite Element Method (FEM)
  - → Divides the domain into smaller, non-overlapping elements.
  - → Suitable for complex geometries and varying boundary conditions.
  - → Often used for structural mechanics, fluid dynamics, and electromagnetic problems.
- Finite Volume Method (FVM)
  - → Integrates over control volumes to ensure conservation laws are satisfied.
  - → Widely used in computational fluid dynamics (CFD).
  - → Balances fluxes between cells, ensuring mass and energy conservation.
- Discontinuous Galerkin Method (DGM)
  - → A finite element method characterized by using discontinuous polynomial approximations within elements.
  - → Combines aspects of both finite element and finite volume methods
  - → Well-suited for solving hyperbolic and convection-dominated PDEs.

FEM

The Finite Element Method (FEM) is a general and flexible method used to solve a wide range of PDEs. It is commonly applied to structural mechanics, heat transfer, and other problems with complex geometries.

- Mathematical Approach :
  - → Uses piecewise polyn. functions that are C^0 across element boundaries.
  - → Minimizes the residual of the PDE using a weighted integral approach.
- Advantages :
  - → Versatility : Can handle complex geometries and bound. conditions with ease.
  - → Theoret. foundation : Strong converg. prop. and error estimation techn.
  - → Adaptive Refinement : Supports both mesh refinement (h-adaptivity) and polynomial enrichment (p-adaptivity).
  - → Applicable to a Wide Range of pbs : Effective for elliptic and parabolic PDEs.
- Disadvantages :
  - → Computational Cost : More complex linear systems compared to simpler methods like finite differences.
  - → Stabilization Required for Hyperbolic Problems : Standard FEM struggles with convection-dominated flows, requiring stabilization techniques.
  - → Continuity Constraints : C^0 continuity across elements, which can limit flexibility in some applications.

The Finite Volume Method (FVM) is primarily used in fluid dynamics and other fields requiring conservation properties. It discretizes the integral form of the PDE over control volumes and computes fluxes across cell bound., ensuring loc. and glob. conservation.

- Mathematical Approach :
  - → Integrates the governing equations over control volumes and uses approximations for fluxes at cell interfaces.
  - → Ensures conservation properties through flux balancing across shared faces.
- Advantages :
  - → Local and Global Conservation : Guarantees that the discretized equations conserve fluxes, making it ideal for fluid and transport problems.
  - → Geometric Flexibility : handle complex geometries using unstruct. meshes.
  - → Handling Discontinuities : Manages shocks and discontinuities better than FEM due to its conservative nature.
- Disadvantages :
  - → Lower-Order Accuracy : Generally, FVM is lower-order unless more complex high-order reconstruction techniques are used.
  - → Extension to High Order is Non-Trivial : achieving high-order accuracy requires additional complexity, such as using higher-order flux reconstruction.
  - → Stab. and Num. Diff. : can suffer from numerical diffusion and stability issues

DGM

The DGM is a hybrid approach combining ideas from both FEM and FVMs. It uses discontinuous polynomial approximations within elements and solves weak forms of the governing equations, which can be applied to various PDEs.

- Mathematical Approach :
  - → Uses piecew. polyn. spaces with no continuity requirement across elements.
  - → Solves a weak form of the PDE within each element and uses numerical fluxes to connect elements.
- Advantages :
  - → High-Order Accuracy : Capable of achieving very high accuracy with high-degree polynomials.
  - → Local Conservation : Ensures local conservation properties, which are critical for mass, momentum, and energy conservation.
  - → Flexibility : hp-adaptivity.
  - → Parallel Efficiency : highly parallelizable.
- Disadvantages :
  - → Complex Implementation : Requires careful handling of inter-element fluxes and boundary conditions.
  - → High comput. Cost : DG has more degrees of freedom per element compared to other methods, leading to higher memory and comput. requirements.

- Discontinuous Galerkin Method (DGM) is ideal for problems requiring high-order accuracy and strict local conservation but comes at a high computational cost.
- Finite Element Method (FEM) is versatile and effective for a broad range of applications but requires stabilization for hyperbolic PDEs.
- Finite Volume Method (FVM) is robust for conservation laws and fluid dynamics but is often limited in accuracy unless extended to higher-order schemes.

The choice of method should be guided by the nature of the PDE, the desired accuracy, and computational resources.

LECTURE 3 :
Introduction to DGM for conservation laws

Outline of the Lecture

Outline

- Time-dependent conservation laws : use of a class of high order nonlinearly stable Runge-Kutta time discretizations.
- Based on a convex combination of first-order forward Euler steps
- Strong stability properties in any semi-norm (total variation semi-norm, maximum norm, entropy condition, etc.) of the forward Euler step.
- Consequence : one only needs to prove nonlinear stability for the first order forward Euler step (easy in many situations) and yields automatically to the same strong stability property for the higher order time discretizations [9].

- Most popular scheme is the following Runge-Kutta method of order 3 (RK3) for solving

$$u_t = L(u,t)$$

where $L(u,t)$ is a spatial discretization operator (not necessarily linear) :

$$\begin{aligned} u^{(1)} &= u^n + \Delta t\, L(u^n, t^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t\, L(u^{(1)}, t^n + \Delta t) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t\, L\left(u^{(2)}, t^n + \frac{1}{2}\Delta t\right) \end{aligned}$$

where $u^n \approx u(t^n)$ where $t_n = n\Delta t$ with $\Delta t > 0$, $n \in \mathbb{N}$

- PDEs that contain high-order spatial derivatives with large coefficients or large propagation speed : RKMs suffer from severe time-step restrictions.
- It is an important and active research subject to study efficient time discretization [10] in DG framework.
- We will not further discuss this important issue through these lectures and we focus to RK1.
- example : for scalar conservation laws $u_t + f(u)_x = 0$, the time step restriction is a function of $f'(u)$ and of the degree of the polynomial approximation.

Outline

Let us consider the following one-dimensional problem

$$PB1 : \begin{cases} u_t + f(u)_x & = & 0, & x \in]0,1[\\ u(x,0) & = & u_0(x), & x \in [0,1] \\ \text{BCs} & \text{on} & x = 0 & \\ \text{BCs} & \text{on} & x = 1 & \end{cases}$$

- Let us consider the following mesh to cover the computational domain $[0, 1]$, consisting of cells $I_i = [x_i, x_{i+1}]$ for $0 \leq i \leq N$ where

$$0 = x_0 < x_1 < \cdots < x_{N+1} = 1$$

- For all i, we define the space-step $h_i = x_{i+1} - x_i$ and $h = \max_i h_i$.
- We assume that the mesh is regular :

$$\exists c > 0; h_i \geq ch$$

- We define a finite element space consisting of piecewise polynomials :

$$V_h^k = \left\{ v; v_{|I_i} \in \mathbb{P}_k(I_i),\ 0 \leq i \leq N \right\}$$

where $\mathbb{P}_k(I_i)$ is the set of polynomials of degree up to k defined on the cell I_i.

- Semi-discrete DGM for solving PB1 is PDB1 :

Find the unique function $u_h = u_h(t) \in V_h^k$ such that $\forall 0 \leq i \leq N$

$$\int_{I_i} (u_h)_t v_h \ dx - \int_{I_i} f(u_h)(v_h)_x \ dx + \hat{f}_{i+1} v_h(x_{i+1}^-) - \hat{f}_i v_h(x_i^+) = 0$$

- $v_h \in V_h^k$ is an arbitrary test function
- $v_h(x_i^{\pm}) = \lim\limits_{\varepsilon \to 0^{\pm}} v_h(x_i \pm \varepsilon)$
- $\hat{f}_i = \hat{f}(u_h(x_i^-, t), u_h(x_i^+, t))$ is the numerical fluxes satisfying
  - → consistency property $\hat{f}(u,u) = f(u)$ (see M. Parisot's lecture),
  - → continuity ($\hat{f}$ is Lipschitz continuous for both arguments,
  - → Monotonicity $\hat{f}(\uparrow, \downarrow)$, for instance, Lax-Friedrichs flux :

$$\hat{f}(u,v) = \frac{1}{2}\left(f(u) + f(v) - c(v-u)\right) \text{ with } c = \max_u |f'(u)|$$

- We look $u_h(x,t) \in V_h^k$ as follows

$$u_h(x,t) = \sum_{i=0}^{N} {\Phi^i}^T(x) U^i(t)$$

→ $U^i(t) = (U_0^i(t), U_1^i(t), \ldots, U_k^i(t))^T$ is the degree of freedom (dof)
→ $\Phi^i(x) = (\phi_0^i(x), \phi_1^i(x), \ldots, \phi_k^i(x))^t$ with ϕ_j^i is a basis function with $\forall i,\ \forall j,\ \phi_j^i(x) \neq 0$ if $x \in I_i$ and 0 otherwise.
→ Many choices exist for the basis of the polynomial, monomial, Lagrange, Dubiner, or Legendre basis [7]
→ Example : monomial basis for $\mathbb{P}_k(I_i)$ functions translated from the interval $(-1,1)$

$$\phi_j^i(x) = \phi_j^b\left(\frac{2(x - x_{i+1/2})}{h_i}\right) \text{ with } x_{i+1/2} = \frac{x_i + x_{i+1}}{2}$$

$$\text{and } \phi_j^b(x) = x^j,\ \Phi^i(x) = \Phi^b\left(\frac{2(x - x_{i+1/2})}{h_i}\right)$$

For instance,

- define $u_{h|I_i}(x,t) = {\Phi^i}^T(x)U^i(t)$

  $\rightarrow \displaystyle\int_{I_i}(u_h)_t v_h\ dx - \int_{I_i} f(u_h)(v_h)_x\ dx + \hat{f}_{i+1}v_h(x_{i+1}^-) - \hat{f}_i v_h(x_i^+) = 0$
  gives $\forall i = 0,\ldots,N,\ \forall V^i$:

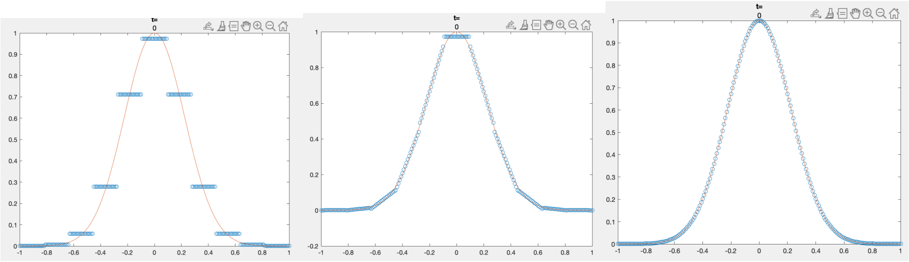
$$ {V^i}^T A \frac{d}{dt}U^i(t) = {V^i}^T\left[\int_{I_i} f(u_{h|I_i}){\Phi^i}_x\ dx - \hat{f}_{i+1}\Phi^i(x_{i+1}) + \hat{f}_i\Phi^i(x_i)\right] $$

where $A = \displaystyle\int_{I_i}\Phi^i(x){\Phi^i}^T(x)\ dx = \frac{h}{2}\int_{-1}^{1}\Phi^b(X){\Phi^b}^T(X)\ dX =$
$\displaystyle\frac{h}{2}\left(\int_{-1}^{1}\phi_i^b(X)\phi_j^b(X)\ dX\right)$,
$\hat{f}_i = f(u_h(x_i^-,t),u_h(x_i^+,t)$ and $\hat{f}_{i+1} = f(u_h(x_{i+1}^-,t),u_h(x_{i+1}^+,t)$ with

$$ u_h(x_i^-,t) = u_{h|I_{i-1}}(x_i,t),\ u_h(x_i^+,t) = u_{h|I_i}(x_i,t), $$

$$ u_h(x_{i+1}^-,t) = u_{h|I_i}(x_{i+1},t) \text{ and } u_h(x_{i+1}^+,t) = u_{h|I_{i+1}}(x_{i+1},t) $$

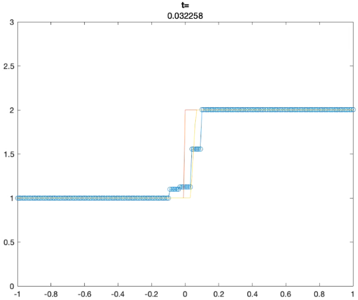
- Solve
$$\forall i = 0\ldots,N,\ \ AU^i = \left[\int_{I_i} \Phi^i(x){\Phi^i}^T(x)\ dx\right] U^{i,0} = \int_{I_i} u_0(x,0)\Phi^i(x)\ dx$$
- Algorithm : for $i = 0,\ldots,N$, do
  - $\rightarrow$ Define $I = (0,1,\ldots,k)$,
  - $\rightarrow$ compute $B = \dfrac{h}{2}\displaystyle\int_{-1}^{1} u0\left(Xh/2 + x_i + \dfrac{h}{2}\right)\Phi^b(X)dX$
  - $\rightarrow$ $U^0(I) = A^{-1}B$
  - $\rightarrow$ define $\mathbb{1} = k+1$
  - $\rightarrow$ $I = I + \mathbb{1}$



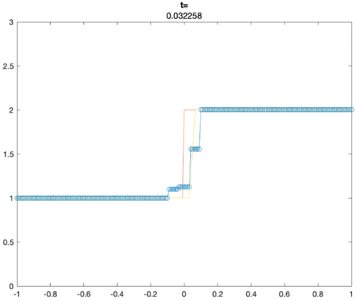
- do while $t = 0, \ldots, T$
- compute Δt^n as $\Delta t^n = \frac{\text{CFL}^{k,n}}{2k+1} \frac{h_i}{\max_i \max_{x \in I_i} |f'(u_h(x, t^n))|}$ [a]
- compute $U^{i,n+1}$
  - $\rightarrow$ Define $I = (0, 1, \ldots, k)$ and $\mathbb{1} = k + 1$.
  - $\rightarrow$ left boundary condition $U^{n+1}(I) = U^{n+1}(I + \mathbb{1})$ [b]
  - $\rightarrow$ compute for $i = 1, \ldots, N - 1$
    - $\Rightarrow$ B(0)=0
    - $\Rightarrow$ flux : $B(1:k) = \frac{h}{2} \int_{-1}^{1} u_{h\,I_i} \left(Xh/2 + x_i + \frac{h}{2}, t^n \right) (\Phi^b)'(X) dX$

      where $u_{h\,I_i}(Xh/2 + x_i + \frac{h}{2}, t^n) = U^n(I)\Phi^b(X)$
    - $\Rightarrow$ numerical flux NumFlux $=$
      $f(u_{h\,I_i}(x_{i+1}, t^n), u_{h\,I_{i+1}}(x_{i+1}, t^n) - f(u_{h\,I_{i-1}}(x_i, t^n), u_{h\,I_i}(x_i, t^n)$
      with, pay attention, $I_{i\pm1}$ refers to the index $I \pm \mathbb{1}$ for the vector U.
  - $\rightarrow$ $U^{n+1}(I) = U^n(I) + \Delta t^n \, (B - \text{NumFlux})$
  - $\rightarrow$ $I = I + \mathbb{1}$
- right boundary condition $U^{n+1}(\text{dof} - \mathbb{1}, \ldots, \text{dof}) = U^{n+1}(\text{dof} - \mathbb{1}, \ldots, \text{dof})$

Let us consider $u_0(x) = 1$ if $x < 0$ and $u_0(x) = 2$ otherwise.

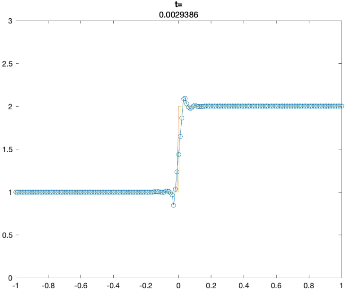


CFL=1.1, N=30

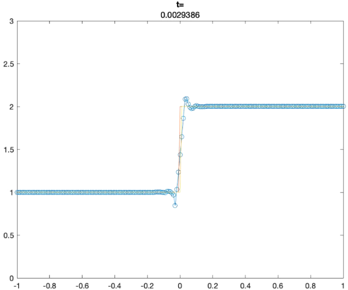


CFL = 0.95, N=30

Let us consider $u_0(x) = 1$ if $x < 0$ and $u_0(x) = 2$ otherwise.



CFL=0.5, N=30



CFL = 0.25, N=30

Outline

- $CFL^{k,n} > 1$ yields to unstable numerical solution $\rightarrow$ in-cell entropy inequality and L^2 stability
- Increasing k yields to spurious oscillations $\rightarrow$ limiters

Entropy inequality

- It is well-known that weak solutions of conservation laws may not be unique.
- The unique physical relevant weak solution (called entropy solution) satisfies the entropy inequality, in the distribution sense :

$$U(u)_t + F(u)_x \leq 0$$

, for any convex entropy $U(u)$ $(U''(u) \geq 0)$ with $F(u) = \int^u U'(u) f'(u)\, du$.

- For a given numerical scheme, is the numerical solution is the entropy one ?
  - $\rightarrow$ quite difficult to show for FDM, FEM and FVM especially for high order scheme
  - $\rightarrow$ DGM is quite easy to show it ! and in particular we have

Theorem

Let u_h be the semi-discrete solution of the DG scheme. Then, one has for all k, for all i, the following entropy inequality

$$\frac{d}{dt}\int_{I_i} U(u_h)\, dx + \hat{F}_{i+1} - \hat{F}_i \leq 0$$

with $U(u) = \dfrac{u^2}{2}$ and some consistent entropy flux $\hat{F}_i = \hat{F}(u_h(x_i^-, t), u_h(x_i^+, t))$.

Let us define

$$B_i(u_h, v_h) = \int_{I_i} (u_h)_t v_h \, dx - \int_{I_i} f(u_h)(v_h)_x \, dx + \hat{f}_{i+1} v_h(x_{i+1}^-) - \hat{f}_i v_h(x_i^+)$$

- take $v_h = u_h$
- denote $\tilde{F}(u) = \int^u f(u) \, du$

Then, $B_i(u_h, v_h) =$
$\int_{I_i} U(u_h)_t \, dx - \tilde{F}(u_h(x_{i+1}^-)) + \tilde{F}(u_h(x_i^+)) + f_{i+1} u_h(x_{i+1}^-) - \hat{f}_i u_h(x_i^+) = 0$ or equivalently

$$B_i(u_h, v_h) = \int_{I_i} U(u_h)_t \, dx + \hat{F}_{i+1} - \hat{F}_i + W_i = 0$$

with $\hat{F}_{i+1} = -\tilde{F}(u_h(x_{i+1}^-)) + \hat{f}_{i+1} u_h(x_{i+1}^-)$ and

$$W_i = -\tilde{F}(u_h(x_i^-)) + \hat{f}_i u_h(x_i^-) + \tilde{F}(u_h(x_i^+)) - \hat{f}_i u_h(x_i^+)$$

$$= (u_h(x_i^+) - u_h(x_i^-)) \left(\tilde{F}'(\xi) - \hat{f}_i) \geq 0 \right)$$

Theorem

Let u_h be the semi-discrete solution of the DG scheme. For compactly supported initial data or periodic boundary conditions, one has

$$\frac{d}{dt}\int_0^1 (u_h(x,t))^2 \, dx \leq 0$$

This is a straightforward consequence of the entropy inequality : we sum up the cell entropy inequality over i. The flux terms telescope and no boundary term is left because of the periodic or compact supported boundary condition.

- For discontinuous solutions, the cell entropy inequality, and the L^2 stability, although helpful, are not enough to control spurious numerical oscillations near discontinuities.
- needs to apply nonlinear limiters to control spurious oscillations to obtain total variation stability
- Starting with a preliminary solution $u_h^{n,\mathrm{pre}} \in V_h^k$, satisfying :

$$\int_{I_i} \frac{u_h^{n+1,\mathrm{pre}} - u_h^{n,\mathrm{pre}}}{\Delta t^n} v_h \, dx - \int_{I_i} f(u_h^{n,\mathrm{pre}})(v_h)_x \, dx + \hat{f}_{i+1}^n v_h(x_{i+1}^-) - \hat{f}_i^n v_h(x_i^+) = 0$$

at time t^n, we want to "limit" spurious oscillations by computing $u_h^n \in V_h^k$ as follows :

$\rightarrow$ $\forall i = 0, \ldots, N, \ \frac{1}{h_i} \int_{I_i} u_h^n(x) \, dx = \frac{1}{h_i} \int_{I_i} u_h^{n,\mathrm{pre}}(x) \, dx$

$\rightarrow$ In region where u is smooth, we should have $u_h^n(x) = u_h^{n,\mathrm{pre}}$.

- Many references [3, 4, 5, 6] for instance.

Outline

Outline

We have presented

- existing classical numerical scheme
- apply the RKDG scheme for scalar conservation laws

To do,

- Introduction to the DG method for parabolic-elliptic equation
- Application of the DG method for a convection-diffusion equation

Outline

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