Modélisation d'écoulements des fluides et environnement

Free surface and groundwater flows modeling

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We have presented

- review of existing classical numerical scheme
- apply the RKDG scheme for scalar conservation laws

To do,

- Introduction to the DG method for parabolic-elliptic equation
- Application of the DG method for a convection-diffusion equation

Elliptic PDEs arise in steady-state problems such as electrostatics, structural analysis, and incompressible fluid flow (e.g., the Poisson equation). Several DG methods have been developed for elliptic problems :

- Interior Penalty DG Methods (IPDG) : One of the most common approaches, IPDG, introduces a penalty term to enforce continuity weakly between the discontinuous elements. The penalty term controls the inter-element jump and ensures stability.
- Local Discontinuous Galerkin (LDG) Methods : LDG methods introduce auxiliary variables to split second-order elliptic equations into first-order systems, which are then discretized using a DG framework. LDG offers advantages in handling higher-order PDEs and flux terms.
- Nonsymmetric DG (NDG) Methods : These methods avoid symmetric treatment of fluxes and often introduce additional stabilizing terms to ensure convergence and robustness for elliptic problems.

Theoretical results for elliptic PDEs, such as error estimates and stability analyses, have been well established, with the method shown to be stable and convergent under appropriate choices of the penalty parameter and element polynomial degrees.

LECTURE 4 :
Introduction to DGM for elliptic/parabolic equations

Let us consider the following two-point boundary problem P on the unit interval

$$\begin{aligned} -(K(x)u'(x))' &= f(x),\ \forall x \in]0,1[\\ u(0) &= 1 \\ u(1) &= 0 \end{aligned}$$

where $K \in C^1(0,1)$ and $f \in C^0(0,1)$. We also assume that

$$\forall x \in [0,1],\ 0 < K_0 \leq K(x) \leq K_1\ .$$

We say that u is a solution of this problem if $u \in C^2(0,1)$ and satisfies the equation pointwise.

Outline

Outline

- Let $x_i = ih$, $i = 0, \dots, N+1$ with $h = \dfrac{1}{N+1}$ for the sake of simplicity.
- As done in the previous lecture, we recall $V_h^k = \{v; v_{I_i} \in \mathbb{P}_k(I_i)\}$ the space of piecewise discontinuous polynomials of degree k on the interval I_i
- we recall that
  - $\rightarrow$ $v(x_i^{\pm}) = \lim_{\varepsilon \to 0} v(x_i \pm \varepsilon)$
  - $\rightarrow$ jump : $[\![v(x_i)]\!] = v(x_i^-) - v(x_i^+)$, $i = 1, N$ and $[\![v(x_0)]\!] = -v(x_0^+)$, $[\![v(x_{N+1})]\!] = v(x_{N+1}^-)$
  - $\rightarrow$ average : $\{v(x_i)\} = \dfrac{1}{2}\big(v(x_i^-) + v(x_i^+)\big)$, $i = 1, N$ and $\{v(x_0)\} = v(x_0^+)$, $\{v(x_{N+1})\} = v(x_{N+1}^-)$
  - $\rightarrow$ jump penalization : $J_0(v, w) = \displaystyle\sum_{i=0}^{N+1} \frac{\sigma^0}{h} [\![v(x_i)]\!][\![w(x_i)]\!]$

- Let $v \in V_h^k = \{v; v_{I_i} \in \mathbb{P}_k(I_i)\}$, multiply equation by and integrate by parts on each interval I_i gives :

$$\int_{I_i} K(x)u'(x)v'(x)\ dx - K(x_{i+1})u'(x_{i+1})v(x_{i+1}^-) + K(x_i)u'(x_i)v(x_i^+)$$

$$= \int_{I_i} f(x)v(x)\ dx,\ \ i = 0, \dots N$$

- By adding all equations above, we get

$$\sum_{i=0}^{N} \int_{I_i} K(x)u'(x)v'(x)\ dx - \sum_{i=0}^{N+1} [\![K(x_i)u'(x_i)v(x_i)]\!] = \sum_{i=0}^{N} \int_{I_i} f(x)v(x)\ dx$$

- Easy to check that $[\![uv]\!] = [\![u]\!]\,\{v\} + [\![v]\!]\,\{u\}$
- Since $[\![K(x_i)u'(x_i)]\!] = 0$, applying this property provides

$$\sum_{i=0}^{N} \int_{I_i} K(x)u'(x)v'(x)\ dx - \sum_{i=0}^{N+1} \{K(x_i)u'(x_i)\}\,[\![v(x_i)]\!] + J_0(u,v)$$
$$= \sum_{i=0}^{N} \int_{I_i} f(x)v(x)\ dx$$

- Since u is continuous : $[\![u(x_i)]\!] = 0$, for $i = 1, \ldots, N$
- if u is a solution of P then u satisfies

$$\sum_{i=0}^{N} \int_{I_i} K(x)u'(x)v'(x)\ dx$$

$$- \sum_{i=0}^{N+1} \{K(x_i)u'(x_i)\} [\![v(x_i)]\!] + \epsilon \sum_{i=0}^{N+1} \{K(x_i)v'(x_i)\} [\![u(x_i)]\!] + J_0(u, v)$$

$$= \sum_{i=0}^{N} \int_{I_i} f(x)v(x)\ dx - \epsilon K(x_0)v'(x_0)u(x_0) + \epsilon K(x_{N+1})v'(x_{N+1})u(x_{N+1})$$

where $\epsilon \in \mathbb{R}$. We restrict to $\epsilon \in \{-1, 0, 1\}$.

Weak DG formulations II

Let $a_\epsilon : V_h^k \times V_h^k \to \mathbb{R}$:

$$a_\epsilon(u,v) = \sum_{i=0}^{N} \int_{I_i} K(x)u'(x)v'(x)\,dx - \sum_{i=0}^{N+1} \{K(x_i)u'(x_i)\} [\![v(x_i)]\!]$$

$$+\epsilon \sum_{i=0}^{N+1} \{K(x_i)v'(x_i)\} [\![u(x_i)]\!] + \frac{\sigma^0}{h} \sum_{i=0}^{N+1} [\![u(x_i)]\!][\![v(x_i)]\!]$$

- if $\epsilon = -1$, then $a_\epsilon(u,v) = a_\epsilon(v,u)$ is symmetric
- if $\epsilon \in \{0,1\}$, then a_ϵ is nonsymmetric

yields to several DGM. More precisely, one has the class of well-known DGM :

- if $\epsilon = -1$ and $\sigma^0 > 0$: Symmetric Interior Penalty Galerkin (SIPG) [1]
- if $\epsilon = -1$ and $\sigma^0 = 0$: Global Elemen Method (GEM) [2]
- if $\epsilon = 1$ and $\sigma^0 = 1$: Nonsymmetric Interior Penalty Galerkin (NIPG) [6]
- if $\epsilon = 1$ and $\sigma^0 = 0$: Nonsymmetric Interior Penalty Galerkin (NIPG) by [3]
- if $\epsilon = 0$: Incomplete Interior Penalty Galerkin (IIPG) [1]
- if $\epsilon = 0$ and $\sigma^0 = 0$: the method is not convergent and not stable ! One cannot even prove the existence and uniqueness.

In practice, it is useful to use variable penalty parameter σ_i^0.

Outline

Since the problem is finite-dimensional, the existence of a solution is equivalent to uniqueness. Let us assume that u_1 and u_2 are two solutions and let us define $w = u_1 - u_2$. Since both u_i, for $i = 1, 2$ satisfies

$$a_\epsilon(u, v) = L(v)$$

for all $v \in V_h^k$ with

$$L(v) = \sum_{i=0}^{N} \int_{I_i} f(x)v(x)\ dx - \epsilon K(x_0)v'(x_0)u(x_0) + \epsilon K(x_{N+1})v'(x_{N+1})u(x_{N+1})$$

It yields to

$$a_\epsilon(w, v) = 0\ .$$

For the NIPG case with $\sigma^0 > 0$, with $v = w$, one has $\forall i$

$$\int_{I_i} K(x)(w'(x))^2\ dx = 0 \text{ and } \frac{\sigma^0}{h}[\![w(x_i)]\!]^2 = 0\ .$$

Since $K > 0$, the first equation implies that w is a constant and the second one implies that the constant is precisely 0 which ends the uniqueness. Proofs in the other case are more complex and we refer to [4] and [5].

Outline

As done in the previous lecture, we consider the monomial basis $\{1, x, x^2, \dots, x^k\}$ which are given by $\phi_j^i(x) = \left(\frac{2}{h}(x - x_{i+1/2})\right)^j$, for $j = 0, \dots, k$, translated from the interval $[-1, 1]$ where $x_{i+1/2}$ is the midpoint of the interval I_i. We expand the DG solution of the problem P as follows :

$$u_h(x) = \sum_{i=0}^{N} {\Phi^i}^T U^i$$

where $U^i \in \mathbb{R}^{k+1}$ are the dof of u_h on the interval I_i.

Let us define

- $A = \int_{I_i} \Phi^i(x)(\Phi^i)^T \, dx$
- $B^{i,j}(x) = \Phi^i(x)(\Phi^j)^T$
- $D_u^{i,j}(x) = (\Phi^i(x))((\Phi^j(x))')^T$
- $D_v^{i,j}(x) = (\Phi^i(x))'((\Phi^j(x)))^T$

and remark that

- $$\{K(x_i)u'(x_i)\} [\![v(x_i)]\!]$$
$$= (V^{i-1})^T \left(\frac{1}{2}K(x_i^-)\Phi^{i-1}(x_i)((\Phi^{i-1})'(x_i))^T U^{i-1} + \frac{1}{2}K(x_i^+)\Phi^{i-1}(x_i)((\Phi^i)'(x_i))^T U\right.$$
$$-(V^i)^T \left(\frac{1}{2}K(x_i^-)\Phi^i(x_i)((\Phi^{i-1})'(x_i))^T U^{i-1} + \frac{1}{2}K(x_i^+)\Phi^i(x_i)((\Phi^i)'(x_i))^T U^i\right)$$
$$= (V^{i-1})^T \left(\frac{1}{2}K(x_i^-)D_u^{i-1,i-1}(x_i)U^{i-1} + \frac{1}{2}K(x_i^+)D_u^{i-1,i}(x_i)U^i\right)$$
$$-(V^i)^T \left(\frac{1}{2}K(x_i^-)D_u^{i,i-1}(x_i)U^{i-1} + \frac{1}{2}K(x_i^+)D_u^{i,i}(x_i)U^i\right)$$

Settings

and

- $$\{K(x_i)v'(x_i)\}\,[\![u(x_i)]\!]$$
$$= (V^{i-1})^T\left(\frac{1}{2}K(x_i^-)D_v^{i-1,i-1}(x_i)U^{i-1} - \frac{1}{2}K(x_i^-)D_v^{i-1,i}(x_i)U^i\right)$$
$$-(V^i)^T\left(\frac{1}{2}K(x_i^+)D_v^{i,i-1}(x_i)U^{i-1} - \frac{1}{2}K(x_i^+)D_v^{i,i}(x_i)U^i\right)$$

- $$[\![u(x_i)]\!][\![v(x_i)]\!] = (u(x_i^-) - u(x_i^+))(v(x_i^-) - v(x_i^+))$$
$$= u(x_i^-)v(x_i^-) - u(x_i^+)v(x_i^-) - u(x_i^-)v(x_i^+) + u(x_i^+)v(x_i^+)$$
$$= (V^{i-1})^T B^{i-1,i-1}(x_i)U^{i-1} - (V^{i-1})^T B^{i-1,i}(x_i)U^i$$
$$-(V^i)^T B^{i,i-1}(x_i)U^{i-1} + (V^i)^T B^{i,i}(x_i)U^i$$

Settings

$$\sum_{i=0}^{N} \int_{I_i} K(x)u'(x)v'(x)\ dx$$

$$- \sum_{i=0}^{N+1} \{K(x_i)u'(x_i)\} [\![v(x_i)]\!] + \epsilon \sum_{i=0}^{N+1} \{K(x_i)v'(x_i)\} [\![u(x_i)]\!] + \frac{\sigma^0}{h} \sum_{i=0}^{N+1} [\![u(x_i)]\!][\![v(x_i)]\!]$$

$$= \sum_{i=0}^{N} \int_{I_i} f(x)v(x)\ dx - \epsilon K(x_0)v'(x_0)u(x_0) + \epsilon K(x_{N+1})v'(x_{N+1})u(x_{N+1})$$

Settings

$$\sum_{i=0}^{N} \int_{I_i} K(x) u'(x) v'(x) \; dx$$

$$- \sum_{i=1}^{N} \left\{ K(x_i) u'(x_i) \right\} [\![v(x_i)]\!] + \epsilon \sum_{i=1}^{N} \left\{ K(x_i) v'(x_i) \right\} [\![u(x_i)]\!] + \frac{\sigma^0}{h} \sum_{i=1}^{N} [\![u(x_i)]\!] [\![v(x_i)]\!]$$

$$+ \frac{\sigma^0}{h} u(x_0) v(x_0) + \frac{\sigma^0}{h} u(x_{N+1}) v(x_{N+1})$$

$$= \sum_{i=0}^{N} \int_{I_i} f(x) v(x) \; dx + \frac{\sigma^0}{h} u_0 v(x_0^+) + \frac{\sigma^0}{h} u_{N+1} v(x_{N+1}^-)$$

$$- K(x_0^+) u'(x_0^+) v(x_0^+) + K(x_{N+1}^-) u'(x_{N+1}^-) v(x_{N+1}^-)$$

Settings

For $i = 1, \ldots, N$ we have

$$-\left\{K(x_i)u'(x_i)\right\}[\![v(x_i)]\!] + \epsilon\left\{K(x_i)v'(x_i)\right\}[\![u(x_i)]\!] + \frac{\sigma^0}{h}[\![u(x_i)]\!][\![v(x_i)]\!]$$

$$= (V^{i-1})^T\left(M^{i-1,i-1}U^{i-1} + M^{i-1,i}U^i\right) + (V^i)^T\left(M^{i,i-1}U^{i-1} + M^{i,i}U^i\right)$$

$$\begin{pmatrix} V^{i-1} \\ V^i \end{pmatrix}\begin{pmatrix} M^{i-1,i-1} & M^{i-1,i} \\ M^{i,i-1} & M^{i,i} \end{pmatrix}\begin{pmatrix} U^{i-1} \\ U^i \end{pmatrix}$$

with

$$M^{i-1,i-1} = -\frac{1}{2}K(x_i^-)D_u^{i-1,i-1}(x_i) + \frac{\epsilon}{2}K(x_i^-)D_v^{i-1,i-1}(x_i) + \frac{\sigma^0}{h}B^{i-1,i-1}(x_i)$$

$$M^{i-1,i} = \frac{1}{2}K(x_i^+)D_u^{i-1,i}(x_i) - \frac{\epsilon}{2}K(x_i^-)D_v^{i-1,i}(x_i) - \frac{\sigma^0}{h}B^{i-1,i}(x_i)$$

$$M^{i,i-1} = \frac{1}{2}K(x_i^-)D_u^{i,i-1}(x_i) - \frac{\epsilon}{2}K(x_i^+)D_v^{i,i-1}(x_i) - \frac{\sigma^0}{h}B^{i,i-1}(x_i)$$

$$M^{i,i} = \frac{1}{2}K(x_i^+)D_u^{i,i}(x_i) + \frac{\epsilon}{2}K(x_i^+)D_v^{i,i}(x_i) + \frac{\sigma^0}{h}B^{i,i}(x_i)$$

Assembly

$$(V^0)^T(A + B^{0,0}(x_0))U^0$$

$$+\sum_{i=1}^{N}(V^{i-1})^T\left(M^{i-1,i-1}(x_i)U^{i-1} + M^{i-1,i}(x_i)U^i\right)$$

$$+\sum_{i=1}^{N}(V^{i})^T\left(M^{i,i-1}(x_i)U^{i-1} + (A + M^{i,i}(x_i))U^i\right)$$

$$+(V^N)^T(B^{N,N}(x_{N+1}))U^N$$

$$\Longleftrightarrow$$

$$(V^0)^T(A + B^{0,0}(x_0) + M^{0,0}(x_1))U^0 + M^{0,1}(x_1)U^1$$

$$+\sum_{i=1}^{N-1}(V^{i})^T\left(M^{i,i-1}(x_i)U^{i-1} + (A + M^{i,i}(x_i) + M^{i,i}(x_{i+1}))U^i + M^{i,i+1}(x_{i+1})U^{i+1}\right)$$

$$+(V^N)^T\left(M^{N,N-1}(x_N)U^{N-1} + (A + B^{N,N}(x_{N+1}) + M^{N,N}(x_N))U^N\right)$$

yielding to

$$V^T\mathcal{A}U$$

Assembly RHS

$$L(v) = \int_{I_0} f(x)v(x)\ dx + \frac{\sigma^0}{h} u_0 v(x_0^+) - K(x_0^+)u'(x_0^+)v(x_0^+)$$
$$+ \sum_{i=1}^{N-1} \int_{I_i} f(x)v(x)\ dx$$
$$+ \int_{I_N} f(x)v(x)\ dx + \frac{\sigma^0}{h} u_{N+1} v(x_{N+1}^-) + K(x_{N+1}^-)u'(x_{N+1}^-)v(x_{N+1}^-)$$
$$\Longleftrightarrow$$

$$L(v) = (V^0)^T \left(F^0 + \frac{\sigma^0}{h} u_0 \Phi^0(x_0) - K(x_0^+)u'(x_0^+)\Phi^0(x_0) \right)$$
$$+ \sum_{i=1}^{N-1} (V^i)^T F^i$$
$$+ (V^N)^T \left(F^N + \frac{\sigma^0}{h} u_{N+1} \Phi^N(x_{N+1}) + K(x_{N+1}^-)u'(x_{N+1}^-)\Phi^N(x_{N+1}) \right)$$

yielding to

$$V^T \mathcal{F}$$

Outline

One can show that if the exact solution is smooth enough, the numerical error decreases as one increases the number of intervals, i.e., as one decreases the mesh size h. We define the numerical error

$$e_h = u - u_h$$

and we define the energy norm

$$\|e_h\| = \left(\sum_{i=0}^{N} \int_{I_i} K(x)(e_h'(x))^2 \, dx + J_0(e_h, e_h) \right)^{1/2} \text{ and } \|e_h\|_2 = \left(\int_0^1 (e_h(x))^2 \, dx \right)^{1/2}$$

and $\beta_1 = \dfrac{1}{\ln(2)} \ln\left(\dfrac{\|e_h\|}{\|e_{h/2}\|} \right)$ and $\beta_2 = \dfrac{1}{\ln(2)} \ln\left(\dfrac{\|e_h\|_2}{\|e_{h/2}\|_2} \right)$ we get

Convergence rates of primal DG method for uniform meshes in one dimension.

Method	β_1	β_2
NIPG $\sigma^0 \geq 0$	k	$k+1$ if k odd k if k even
SIPG $\sigma^0 > \sigma^0_*$	k	$k+1$
IIPG $\sigma^0 > \sigma^0_*$	k	$k+1$ if k odd k if k even

We have presented

- existing classical numerical scheme
- apply the IIPG scheme for elliptic/parabolic equations

To do,

- Application of the DGM for a convection-diffusion equation

Outline

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