Adaptive scheme based on numerical entropy production for conservation laws.

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Solving a *non linear hyperbolic conservation laws*

$$\begin{cases} \dfrac{\partial \boldsymbol{w}}{\partial t} + \dfrac{\partial \boldsymbol{f}(\boldsymbol{w})}{\partial x} = 0, \ (t,x) \in \mathbf{R}^+ \times \mathbf{R} \\ \boldsymbol{w}(0,x) = \boldsymbol{w}_0(x), \ x \in \mathbf{R} \end{cases} \tag{1}$$

with high accuracy is a *challenging problem* since it is well-known that solutions can and will breakdown at a finite time developing complex structure, even if the initial data are smooth. **As a consequence**, accurate solutions require ***a large number of cells*** which lead to ***time consuming simulations***.

Our aim is to construct a multi scale adaptive numerical scheme which *save the cpu time keeping the order of accuracy* (see [Ers13]).

Adaptive grids based on entropy production

Mathematical entropy : adaptive strategy

The concept of entropy refers to the convex continuous entropy $s(\boldsymbol{w})$ of flux $\psi(\boldsymbol{w})$, for which one has

$$\frac{\partial s(\boldsymbol{w})}{\partial t} + \frac{\partial \psi(\boldsymbol{w})}{\partial x} \begin{cases} \equiv 0 & \text{classical solution} \\ < 0 & \text{weak solution (discontinuous solution)} \end{cases} \text{ with } \psi' = s'f' . \tag{2}$$

It provides an *a posteriori error indicator* and a useful tool to construct automatic mesh refinement.

Finite volume formulation & approximation

A standard Finite Volume discretisation of Equations (1)-(2) lead to

$$\boldsymbol{w}_k^{n+1} = \boldsymbol{w}_k^n - \frac{\delta t_n}{h_k}\left(\boldsymbol{F}_{k+1/2}^n - \boldsymbol{F}_{k-1/2}^n\right) \quad \text{and} \quad S_k^n = \frac{s_k^{n+1} - s_k^n}{\delta t_n} + \frac{\psi_{k+1/2}^n - \psi_{k-1/2}^n}{h_k} ,$$

where

$$\boldsymbol{F}_{k+1/2}(\boldsymbol{w}_k^n, \boldsymbol{w}_{k+1}^n) \approx \frac{1}{\delta t_n}\int_{t_n}^{t_{n+1}} \boldsymbol{f}(\boldsymbol{w}(x_{k\pm 1/2}, s)\, ds \quad \text{and} \quad \psi_{k\pm 1/2} \approx \frac{1}{\delta t_n}\int_{t_n}^{t_{n+1}} \psi(\boldsymbol{w}(x_{k\pm 1/2}, s)\, ds$$

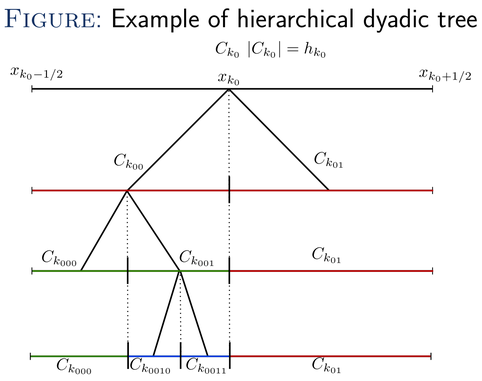
with

$$\boldsymbol{w}_k^n \simeq \frac{1}{h_k}\int_{C_k} \boldsymbol{w}\,(x, t_n)\ dx\ .$$

S_k^n stands for the numerical density of entropy production.

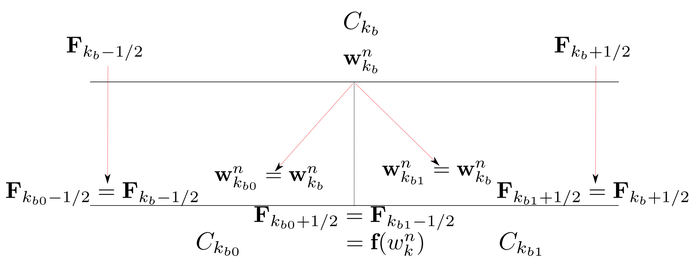
Numerical density of entropy production as a refinement criterion

FIGURE: Example of hierarchical dyadic tree

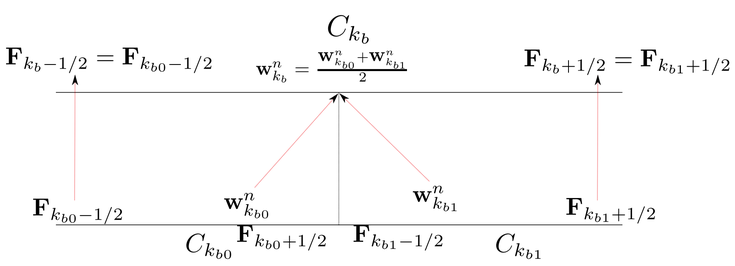


Define
$\alpha_{\max}$: mesh refinement parameter ,
$\alpha_{\min}$: mesh coarsening parameter ,
$\bar{S} = \dfrac{1}{|\Omega|}\sum_{k_b} S_{k_b}^n$: mesh refinement parameter
then for each cell C_{k_b} :

- if $S_{k_b}^n > \bar{S}\alpha_{max}$, the mesh is split into two sub-cells $C_{k_{b0}}$ and $C_{k_{b1}}$:



- if $S_{k_{b0}}^n < \bar{S}\alpha_{min}$ and $S_{k_{b1}}^n < \bar{S}\alpha_{min}$, the mesh is coarsened into a cell C_{k_b} :



Major drawback

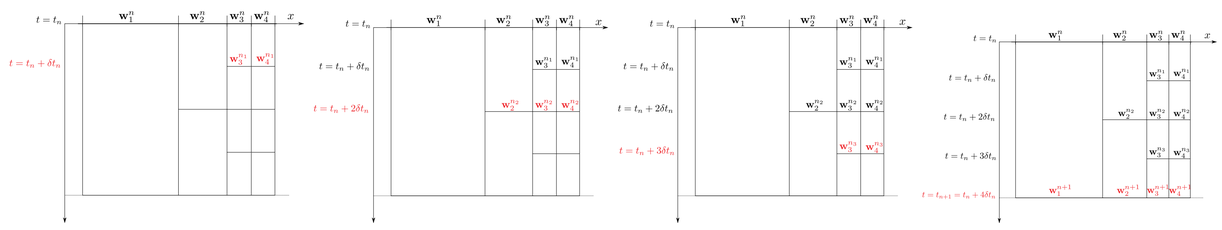
The numerical scheme being stable under a CFL condition

$$\frac{\delta t_n}{\min_k h_k} \max \lambda(D_{\boldsymbol{w}}\boldsymbol{f}) < 1 \Longrightarrow \text{the less } h_k \text{ the less } \delta t_n \Longrightarrow \text{time restriction}$$

One can overpass using local time stepping algorithm and ***save the CPU time***.

Local time stepping algorithm

All cells are sorted in groups, called ***level***.
Following [Alt09], the local time stepping algorithm can be illustrated as follows :



(a) At time $t_n + \delta t_n$ (b) At time $t_n + 2\delta t_n$. (c) At time $t_n + 3\delta t_n$. (d) At time $t_{n+1} := t_n + 4\delta t_n$.

FIGURE: Illustration of a global time advancement with a first order scheme.

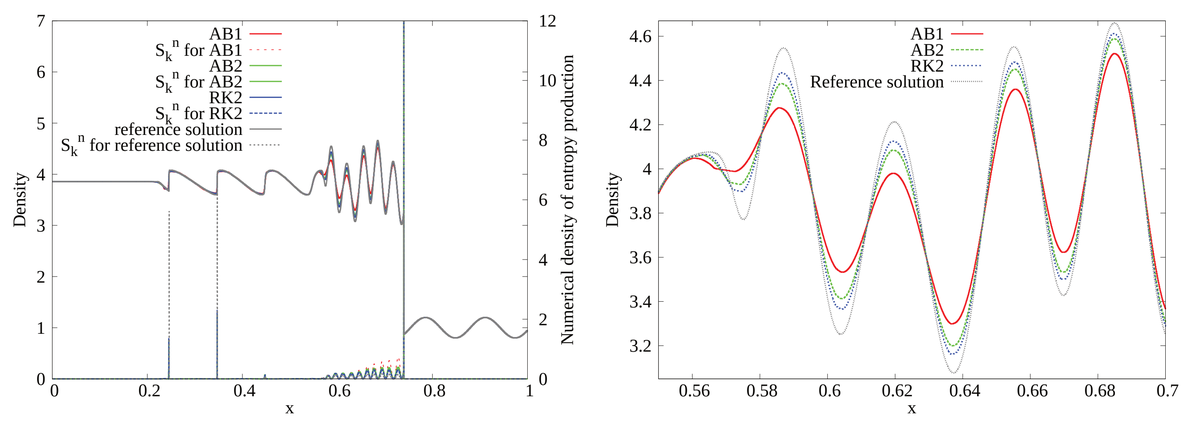
As done in [Ers13], space second order MUSCL reconstruction and Adams-Bashforth high order time integration can be easily implemented.

Numerical experiments

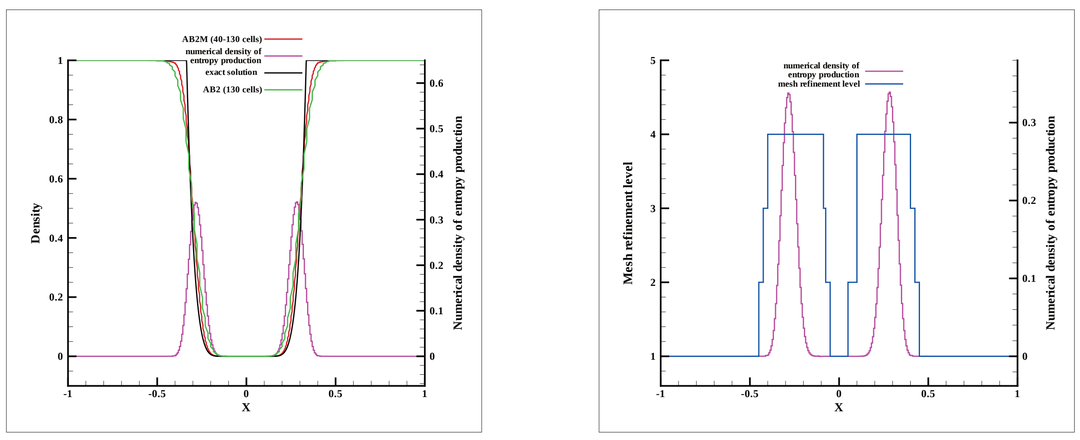
All tests have been performed are computed in the case of the one-dimensional gas dynamics equations for ideal gas with an *Intel(R) Core(TM) i5-2500 CPU @ 3.30GHz*

	$\lvert\rho - \rho_{ex}\rvert_{l_t^1 l_x^1}$ *vs* $N_{L_{\max}}$	$\lvert\rho - \rho_{ex}\rvert_{l_x^1}$ *vs* $N_{L_{\max}}$	$\lvert\rho - \rho_{ex}\rvert_{l_t^1 l_x^1}$ *vs* cpu-time
AB1U	0.66	0.69	0.27
AB1	3.20	3.31	0.66
AB1M	3.55	3.56	0.82

TABLE: Numerical order on the density error using $l_t^1 l_x^1$ norm and l_x^1 at time $t = 0.13$ with respect to the averaged number of cells, and the $l_t^1 l_x^1$ norm on the density error with respect to the cpu-time (in $\log_{10}$ scale).



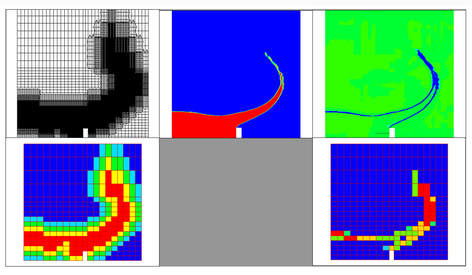
(a) Shu-Osher test case. (b) Shu-Osher test case.



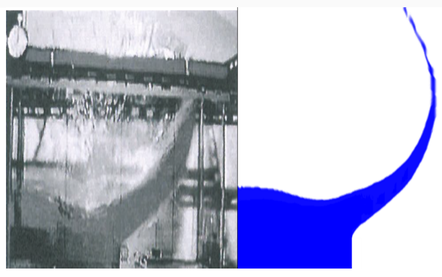
(c) Two rarefaction test case (vacuum test case) (d) Two rarefaction test case (vacuum test case)

Concluding remarks and perspectives

Up to now, all computations can be accelerated with the proposed numerical scheme keeping the same order of accuracy of any numerical solver. The next step is to *improve the accuracy* of the scheme in this framework, especially to well-capture the so-called *contact discontinuities* and prove some convergence rate. The study in the case of non homogeneous system is also in progress as well as the 2D and 3D extensions of the numerical code.



(e) 2D dambreak problem : $\alpha_{\max} = 0.1$, $\alpha_{\max} = 0.2$, $L_{\max} = 5$ using $\bar{S}$ and 321 domains. (top left : mesh, top middle : ρ, top right : S_k^n, bottom left : level, bottom right : $\dfrac{1}{|D|}\displaystyle\int_D S_k^n$)



(f) 2D dambreak problem : confrontation with experimental test

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