Formal derivation of Saint-Venant-Exner-like model:

Vertically averaged Vlasov-Navier-Stokes equations

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Outline of the talk

Outline

Introduction

Formal derivation of the “mixed” CNSEs

Formal derivation of the MENT model
- The non-dimensional “mixed” system
- System vertically averaged

Examples
- Example 1: a viscous sedimentation model
- Example 2: the Grass sedimentation model

Perspective

The Saint-Venant-Exner model

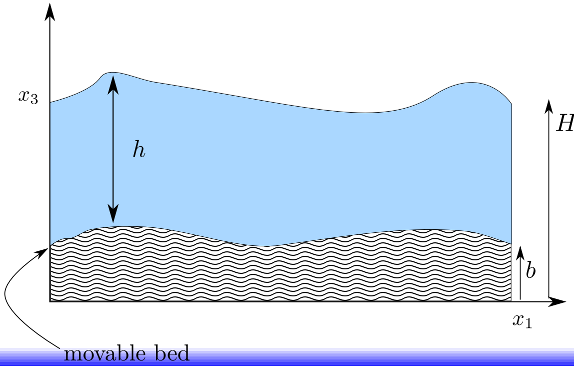
Saint-Venant equations for the hydrodynamic part:

$$\begin{cases} \partial_t h + \mathrm{div}(q) = 0, \\ \partial_t q + \mathrm{div}\left(\dfrac{q \otimes q}{h}\right) + \nabla\left(g\dfrac{h^2}{2}\right) = -gh\nabla b \end{cases} \tag{1}$$

$+$

a bedload transport equation for the morphodynamic part:

$$\partial_t b + \xi \mathrm{div}(q_b(h, q)) = 0 \tag{2}$$



The Saint-Venant-Exner model

Saint-Venant equations for the hydrodynamic part:

$$\begin{cases} \partial_t h + \text{div}(q) = 0, \\ \partial_t q + \text{div}\left(\frac{q \otimes q}{h}\right) + \nabla\left(g\frac{h^2}{2}\right) = -gh\nabla b \end{cases} \quad (1)$$

$$+$$

a bedload transport equation for the morphodynamic part:

$$\partial_t b + \xi \text{div}(q_b(h, q)) = 0 \quad (2)$$

with

- h the water height from the surface $z = b(t, x)$,
- $q = hu$ the water discharge,
- q_b the sediment discharge (given by an empirical law: Grass equation [G81], The Meyer-Peter and Műller equation [MPM48]),
- and $\xi = 1/(1 - \psi)$ the porosity of the sediment layer.

[MPM48] *E. Meyer-Peter and R. Müller*, Formula for bed-load transport, *Rep. 2nd Meet. Int. Assoc. Hydraul. Struct. Res., 39–64*, 1948.

[G81] *A.J. Grass*, Sediment transport by waves and currents, *SERC London Cent. Mar. Technol. Report No. FL29* , 1981.

Outline

Vlasov equation for sediments

$$\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v((F + \vec{g})f) = r\Delta_v f \tag{3}$$

where

- f the density of particles,
- $\vec{g}$ is the gravity vector $(0, 0, -g)^t$, and
- F is the Stokes drag force:

$$F = \frac{6\pi\mu a}{M}(u - v) \quad \begin{array}{l} \text{with } a = cte \text{ the radius,} \\ M = \rho_p \frac{4}{3}\pi a^3 \text{ the mass,} \\ \rho_p \text{ the mass density of sediments,} \\ \text{and } \mu \text{ a characteristic viscosity,} \\ u \text{ velocity of the fluid} \end{array} \tag{4}$$

- $r\Delta_v f$ is the Brownian motion of the particles with $r > 0$ is the velocity of the diffusivity given by the Einstein formula:

$$r = \frac{kT}{M}\frac{6\pi\mu a}{M} = \frac{kT}{M}\frac{9\mu}{2a^2\rho_p} \tag{5}$$

in which k is the Boltzmann constant, $T > 0$ is the temperature of the suspension, assumed constant.

Compressible Navier-Stokes equations

$$
\begin{cases}
\partial_t \rho_w + \mathrm{div}(\rho_w u) = 0, \\
\partial_t(\rho_w \mathbf{u}) + \mathrm{div}_{\mathbf{x}}(\rho_w \mathbf{u} \otimes \mathbf{u}) + \partial_{x_3}(\rho_w \mathbf{u} v) + \nabla_{\mathbf{x}} p(\rho) \\
= \mathrm{div}_{\mathbf{x}}\left(\mu_1(\rho) D_{\mathbf{x}}(\mathbf{u})\right) + \partial_{x_3}\Big(\mu_2(\rho)(\partial_{x_3}\mathbf{u} + \nabla_{\mathbf{x}} u_3)\Big) + \nabla_x(\lambda(\rho)\mathrm{div}(u)) \\
+\mathfrak{F}, \\
\partial_t(\rho_w u_3) + \mathrm{div}_{\mathbf{x}}(\rho_w \mathbf{u} u_3) + \partial_{x_3}(\rho_w u_3^2) + \partial_{x_3} p(\rho) \\
= \mathrm{div}_{\mathbf{x}}\Big(\mu_2(\rho)(\partial_{x_3}\mathbf{u} + \nabla_{\mathbf{x}} u_3)\Big) + \partial_{x_3}(\mu_3(\rho)\partial_{x_3} u_3) + \partial_{x_3}(\lambda(\rho)\mathrm{div}(u)) \\
p = p(t,x) = g\dfrac{h(t,\mathbf{x})}{4\rho_f}\rho^2(t,x)
\end{cases}
\tag{6}
$$

with $u = (\mathbf{u}, u_3)$, $x = (\mathbf{x}, x_3)$ and $\mu_i \neq \mu_j$.

- ρ_w the density of the fluid, ρ_s the macroscopic density of sediments, $\rho = \rho_w + \rho_s$ with $\rho_s = \int_{\mathbb{R}^3} f\,dv$,
- $\mathfrak{F}$ the coupling of fluid-sediment interaction, including the gravity source term:

$$
\mathfrak{F} = -\int_{\mathbb{R}^3} Ffdv + \rho_w \vec{g} \tag{7}
$$

Fluid sediment coupling

$$
\left\{
\begin{array}{l}
\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v\left(\left(\frac{6\pi\mu a}{M}(u - v) + \vec{g}\right) f\right) = \frac{kT}{M}\frac{9\mu}{2a^2\rho_p}\Delta_v f, \\
\\
\partial_t \rho_w + \mathrm{div}(\rho_w u) = 0, \\
\partial_t(\rho_w \mathbf{u}) + \mathrm{div}_{\mathbf{x}}(\rho_w \mathbf{u} \otimes \mathbf{u}) + \partial_{x_3}(\rho_w \mathbf{u} v) + \nabla_{\mathbf{x}} p(\rho) \\
= \mathrm{div}_{\mathbf{x}}\left(\mu_1(\rho) D_{\mathbf{x}}(\mathbf{u})\right) + \partial_{x_3}\left(\mu_2(\rho)(\partial_{x_3}\mathbf{u} + \nabla_{\mathbf{x}} u_3)\right) \\
+\nabla_x(\lambda(\rho)\mathrm{div}(u)) \\
+\mathfrak{F}, \\
\\
\partial_t(\rho_w u_3) + \mathrm{div}_{\mathbf{x}}(\rho_w \mathbf{u} u_3) + \partial_{x_3}(\rho_w u_3^2) + \partial_{x_3} p(\rho) \\
= \mathrm{div}_{\mathbf{x}}\left(\mu_2(\rho)(\partial_{x_3}\mathbf{u} + \nabla_{\mathbf{x}} u_3)\right) + \partial_{x_3}(\mu_3(\rho)\partial_{x_3} u_3) \\
+\partial_{x_3}(\lambda(\rho)\mathrm{div}(u))
\end{array}
\right.
\tag{8}
$$

With boundary conditions:

free surface: a normal stress continuity.

movable bed: a general wall-law condition and continuity of the velocity at the interface $x_3 = b(t, \mathbf{x})$.

kinematic: **???** $\overline{☺}$ replaced with the equation:

$$S = \partial_t b + \sqrt{1 + |\nabla_{\mathbf{x}} b|^2} u_{|x_3=b} \cdot n_b \qquad (9)$$

and $S - \sqrt{1 + |\nabla_{\mathbf{x}} b|^2} u_{|x_3=b} \cdot n_b$ may plays the role of incoming and outgoing particles.

[MSR03] *N. Masmoudi and L. Saint-Raymond*, From the Boltzmann Equation to the Stokes-Fourier System in a Bounded Domain, *Communications on pure and applied mathematics, 53(9):1263–1293*,2003.

Dimensionless number and asymptotic ordering

Let

- $\sqrt{\theta}$ be the fluctuation of kinetic velocity,
- $\mathfrak{U}$ be a characteristic vertical velocity of the fluid,
- $\mathfrak{T}$ be a characteristic time,
- τ be a relaxation time,
- $\mathfrak{L}$ be a characteristic vertical height,

and

$$B = \frac{\sqrt{\theta}}{\mathfrak{U}}, \quad C = \frac{\mathfrak{T}}{\tau}, \quad F = \frac{g\mathfrak{T}}{\sqrt{\theta}}, \quad E = \frac{2}{9}\left(\frac{a}{\mathfrak{L}}\right)^2 \frac{\rho_p}{\rho_f} C \qquad (10)$$

with the following asymptotic regime:

$$\frac{\rho_p}{\rho_f} = O(1), \quad B = O(1), \quad C = \frac{1}{\varepsilon}, \quad F = O(1), \quad E = O(1). \qquad (11)$$

[GJV] *T. Goudon and P-E. Jabin and A. Vasseur*, Hydrodynamic limit for the Vlasov-Navier-Stokes Equations. I. Light particles regime, *Indiana Univ. Math. J., 53(6):1495–1515*,2004.

Approximation at main order with respect to ε

Asymptotic expansion of f, u, p and ρ as: $f = f^0 + \varepsilon f^1 + O(\varepsilon^2)$, ...
Then at order $1/\varepsilon$

$$\begin{cases} \mathrm{div}_v((u^0 - v)f^0 - \nabla_v f^0) = 0, \\ \int_{\mathbb{R}^3} (v - u^0) f^0 \, dv = 0. \end{cases} \tag{12}$$

Let ρ_s and V be the macroscopic density and the macroscopic speed V of particles:

$$\begin{pmatrix} \rho_s \\ \rho_s V \end{pmatrix} = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \end{pmatrix} f \, dv \tag{13}$$

Then Equations (12) provide:

$$f^0 = \frac{1}{(2\pi)^{3/2}} \rho_s^0 \, e^{-\frac{1}{2}\|u^0 - v\|^2} \text{ and } V^0 = u^0. \tag{14}$$

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Approximation at main order with respect to ε

Then at order 1: Integrating Vlasov equation against 1 and v:

$$\begin{cases} \partial_t \rho_s^0 + B\text{div}(\rho_s^0 u^0) = 0 \\ \\ \partial_t(\rho_s^0 u^0) + B\text{div}_x(\rho_s^0\, u^0 \otimes u^0) + B\nabla_x(\rho_s^0) \\ = \int_{\mathbb{R}^3} (u^0 - v) f^1\, dv + \int_{\mathbb{R}^3} u_1 f^0\, dv - F\rho_s^0 \vec{k} \end{cases} \quad (12)$$

On the other hand, the dimensionless CNSEs (Σ being the anisotropic viscous tensor):

$$\begin{cases} \partial_t \rho_w^0 + B\text{div}(\rho_w^0 u^0) = 0, \\ \partial_t(\rho_w^0 u^0) + B\text{div}_x(\rho_w^0 u^0 \otimes u^0) + B\nabla_x p^0 = 2\, E \Big(\text{div}(\Sigma^0 : D(u^0)) \\ +\nabla(\lambda \text{div}(u^0))\Big) + \int_{\mathbb{R}^3} (v - u^0) f^1\, dv - \int_{\mathbb{R}^3} u^1 f^0\, dv - F\rho_w^0 \vec{k}. \end{cases} \quad (13)$$

Adding two system and returning to physical variables, we obtain the "mixed" model:

$$
\left\{
\begin{array}{l}
\partial_t \rho + \mathrm{div}(\rho u) = 0, \\
\partial_t(\rho \mathbf{u}) + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u} \otimes \mathbf{u}) + \partial_{x_3}(\rho \mathbf{u} v) + \nabla_{\mathbf{x}} P \\
= \mathrm{div}_{\mathbf{x}}\left(\mu_1(\rho) D_{\mathbf{x}}(\mathbf{u})\right) + \partial_{x_3}\Big(\mu_2(\rho)(\partial_{x_3}\mathbf{u} + \nabla_{\mathbf{x}} u_3)\Big) \\
+\nabla_x(\lambda(\rho)\mathrm{div}(u)) \\
\\
\partial_t(\rho u_3) + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u} u_3) + \partial_{x_3}(\rho u_3^2) + \partial_{x_3} P \\
= \mathrm{div}_{\mathbf{x}}\Big(\mu_2(\rho)(\partial_{x_3}\mathbf{u} + \nabla_{\mathbf{x}} u_3)\Big) + \partial_{x_3}(\mu_3(\rho)\partial_{x_3} u_3) \\
+\partial_{x_3}(\lambda(\rho)\mathrm{div}(u))
\end{array}
\right.
\tag{14}
$$

where

$$P = p + \theta \rho_s.$$

Outline

Asymptotic analysis: "thin layer"

- Vertical movements are assumed small with respect to horizontal one,
- Vertical length is assumed small with respect to horizontal one,

i.e. we compare:

- $\mathcal{L}$ and L (the characteristic length of the domain),
- $\mathcal{U}$ and U (the characteristic horizontal velocity of the fluid) .

Outline

We introduce a small parameter such as:

$$\varepsilon \approx \frac{\mathcal{L}}{L} \approx \frac{\mathcal{U}}{U}.$$

and

- T such as $T = L/U$,
- $\overline{\rho}$ such as $P = \overline{\rho} U^2$,

$$\widetilde{t} = \frac{t}{T}, \quad \widetilde{x} = \frac{x}{L}, \quad \widetilde{y} = \frac{y}{\mathcal{L}}, \quad \widetilde{u} = \frac{u}{U}, \quad \widetilde{v} = \frac{v}{\mathcal{U}},$$

$$\widetilde{p} = \frac{P}{\overline{p}}, \quad \widetilde{\rho} = \frac{\rho}{\overline{\rho}}, \quad \widetilde{\rho}_s = \frac{\rho_s}{\overline{\rho}}, \quad \widetilde{H} = \frac{H}{\mathcal{L}}, \quad \widetilde{b} = \frac{b}{\mathcal{L}},$$

$$\widetilde{\lambda} = \frac{\lambda}{\overline{\lambda}}, \qquad \widetilde{\mu}_j = \frac{\mu_j}{\overline{\mu}_j}, j = 1, 2, 3.$$

Asymptotic ordering

With

$$\frac{\mu_i(\rho)}{Re_i} = \varepsilon^{i-1}\nu_i(\rho),\ i = 1, 2, 3 \text{ and } \frac{\lambda(\rho)}{Re_\lambda} = \varepsilon^2\gamma(\rho). \tag{15}$$

where

$$F_r = \frac{U}{\sqrt{g\,\mathcal{L}}}, \quad Re_i = \frac{\overline{\rho}UL}{\overline{\mu_i}}, \quad Re_\lambda = \frac{\overline{\rho}UL}{\overline{\lambda}}. \tag{16}$$

is the Froude number F_r, the Reynolds number associated to the viscosity μ_i (i=1,2,3), Re_i and the Reynolds number associated to the viscosity λ, Re_λ .
We also set

$$\overline{S} = \varepsilon U.$$

Hydrostatic approximation

We write the "mixed" system under the non-dimensionnal form with $u = u_0 + \varepsilon u^1$ gives:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho\, u) + \partial_y(\rho v) = 0 \\ \partial_t(\rho\, u) + \mathrm{div}_x(\rho\, u \otimes u) + \partial_y(\rho\, vu) + \frac{1}{F_r^2}\nabla_x p(\rho) = \mathrm{div}_x(\nu_1 D_x(u)) \\ +\partial_y\left(\nu_2 \frac{1}{\varepsilon}\partial_y u^1\right), \\ h(t,x)\rho(t,x,y) = 2(H(t,x) - y) \end{cases} \tag{17}$$

where u_0 is again written as u. Free surface condition:

$$\begin{cases} -\nu_1 D_x(u)\nabla_x H + \left(\nu_2 \frac{1}{\varepsilon}\partial_y u^1\right) = 0 \\ p(\rho) = 0 \end{cases} \tag{18}$$

and bottom condition:

$$\left\{\begin{array}{l} -\nu_1 D_x(u)\nabla_x b + \nu_2 \frac{1}{\varepsilon}\partial_y u^1 = \begin{pmatrix} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u) \end{pmatrix}, \\ \\ \nu_2 \partial_y u \cdot \nabla_x b = 0, \\ \\ \partial_t b + u(t,x,b)\cdot\nabla_x b - v(t,x,b) = \dfrac{\overline{S}}{\varepsilon\, U} S \end{array}\right. \tag{19}$$

Back

On the other hand, we have:

$$\partial_y\left(\nu_2\partial_y u\right) = O(\varepsilon), \quad \left(\nu_2\partial_y u\right)_{|y=H} = O(\varepsilon), \quad \left(\nu_2\partial_y u\right)_{|y=b} = O(\varepsilon).$$

which imply:

$$u(t,x,y) = \overline{u}(t,x) + O(\varepsilon).$$

The mass equation

For any function f, we note the mean value of f over the vertical as

$$h(t,x)\bar{f}(t,x) = \int_b^H f\,dz.$$

Hydrostatic equation $h\rho = 2(H-z) \rightarrow$

$$\int_b^H \rho\,dz = \frac{1}{h}\int_b^H h\rho\,dz = \frac{2}{h}\int_b^H (H-z)\,dz = h. \tag{20}$$

The mean pressure is written as follows:

$$\int_b^H h\rho^2\,dz = \frac{4}{3}h^2. \tag{21}$$

Using

- Leibniz formulas,
- Free surface condition and bottom condition,
- $u = \overline{u} + O(\varepsilon)$,
- Equation (20),

we obtain the averaged mass equation:

$$\partial_t h + \mathrm{div}(h\bar{u}) = 0$$

The momentum equation

Proceeding as before : integrating the horizontal momentum equations for $b \leqslant z \leqslant H$ gives:

$$\partial_t(h\,\overline{u}) + \mathrm{div}_x(h\,\overline{u} \otimes \overline{u}) + \frac{1}{3\,F_r^2}\nabla_x(h^2)$$

$$+\Big(\rho\,u\,(\partial_t b + u \cdot \nabla_x b - w)\Big)_{|z=b}\nabla_x b$$

$$-\Big(\rho\,u\,(\partial_t H + u \cdot \nabla_x H - w)\Big)_{|z=H}\nabla_x H$$

$$= \mathrm{div}_x\Big(\int_b^H D(u-\overline{u})\,dz + \overline{(\nu_1)}hD(\overline{u})\Big)$$

$$+\Big(\frac{\nu_2}{\varepsilon}\partial_z u^1 - \nu_1 D_x(u)\nabla_x b\Big)_{|z=b}$$

$$+\Big(\nu_1 D(\bar{u})\nabla_x H - \frac{\nu_2}{\varepsilon}\partial_z u^1\Big)_{|z=H}$$

- Using boundary conditions ▸ Go on term ▸ Go1,
- $u = \overline{u} + O(\varepsilon)$,
- setting $S = 0$ (for the sake of simplicity),

we finally obtain:

$$\partial_t(h\bar{u}) + \operatorname{div}(h\bar{u} \otimes \bar{u}) + \frac{1}{3\,F_r^2}\nabla h^2 = -\frac{h}{F_r^2}\nabla b + \operatorname{div}(hD(\bar{u})) - \begin{pmatrix} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u) \end{pmatrix} \tag{22}$$

Remark

$S \neq 0$ modify the hydrodynamic part of the flow by adding a source term to the:

- mass equation: $-2S$,
- momentum equations: $-2uS$.

Outline

Although the MENT model is close to SVEEs, we also have, freely, stability and existence result of weak solution

Outline

The viscous model [ZLFN08]

We set $u\nabla_x b - v = \operatorname{div}(\alpha hu\,|u|^k - \beta\nu\nabla b)$ for some α and β satisfying some relation The model:

$$\left\{\begin{array}{lcl} \partial_t h + \operatorname{div}(hu) & = & 0 \\ \partial_t(hu) + \operatorname{div}(hu \otimes u) + gh\nabla\left(\dfrac{h}{3} + b\right) & = & 2\nu\operatorname{div}(hD(u)) \\ \partial_t b + \operatorname{div}(\alpha hu\,|u|^k - \beta\nu I_d\nabla b) & = & 0 \end{array}\right. \tag{23}$$

If

$$L^2(\Omega) \ni h_{|t=0} = h_0 \geqslant 0, \quad b_{|t=0} = b_0 \in L^2(\Omega), \quad hu_{|t=0} = m_0 \tag{24}$$

and

$$|m_0|^2 / h_0 \in L^1(\Omega), \quad \nabla\sqrt{h_0} \in L^2(\Omega)^2 \tag{25}$$

where $\Omega = \mathcal{T}^2$ is the torus.

[ZLFN08] *J-D Zabsonré and C. Lucas and E. Fernández-Nieto*, An Energetically Consistent Viscous Sedimentation Model, *Mathematical Models and Methods in Applied Sciences 19(3):477–499*, 2009.

The stability result

Then the main result presented here, is a straightforward consequence to the one presented in [ZLFN08], is:

Theorem

Let α, β and $\gamma = \gamma(\alpha, \beta)$, $\delta = \delta(\beta)$ (called stability coefficient) such as

$$
\begin{aligned}
& 0 < \beta < 2, \alpha > 0 \\
& \phi(\beta) = \frac{2}{2-\beta} > 0, \\
& \gamma(\alpha, \beta) = 3\alpha\phi(\beta) > 0, \\
& \delta(\beta) = \phi(\beta) - 1 > 0.
\end{aligned}
\tag{26}
$$

The stability result

Then the main result presented here, is a straightforward consequence to the one presented in [ZLFN08], is:

Theorem

Let (h_n, u_n, b_n) be a sequence of weak solutions of (23) with initial conditions (24)-(25), in the following sense: $\forall k \in [0, 1/2]$:

- System (23) holds in $(\mathcal{D}'((0,T)\times\Omega))^4$ with (24)-(25),
- Energy (26), Entropy (28) and the following regularities are satisfied:

$$\begin{array}{ll}
\sqrt{h}u \in L^\infty(0,T;(L^2(\Omega))^2) & \sqrt{h}\nabla u \in L^2(0,T;(L^2(\Omega))^4) \\
h^{1/(k+2)}u \in L^\infty(0,T;(L^{k+2}(\Omega))^2) & h/3 + b \in L^\infty(0,T;L^2(\Omega)), \\
\nabla(h/3) + \nabla b \in L^2(0,T;(L^2(\Omega))^2) & \nabla\sqrt{h} \in L^\infty(0,T;(L^2(\Omega))^2), \\
h^{1/k}D(u)^{2/k} \in L^k(0,T;(L^k(\Omega))^4). &
\end{array}$$

The stability result

Then the main result presented here, is a straightforward consequence to the one presented in [ZLFN08], is:

Theorem

If $h_0^n \geqslant 0$ and the sequence $(h_0^n, u_0^n, m_0^n) \to (h_0, u_0, m_0)$ converges in $L^1(\Omega)$ then, up to a subsequence, the sequence (h_n, u_n, m_n) converges strongly to a weak solution of (23) and satisfy Energy (26), Entropy (28) inequalities.

Outline of the proof

Lemma (Energy)

Let (h, u, b) be a regular solution of (23) and γ, δ satisfying condition (26). Then we have:

$$\frac{d}{dt}\int_\Omega \frac{h\,|u|^2}{2} + \frac{\gamma(\alpha,\beta)}{k+2} h\,|u|^{k+2} + g\phi(\beta)\left(\sqrt{\frac{3}{2}}b + \sqrt{\frac{1}{6}}h\right)^2 + \delta(\beta) h \frac{|\psi|^2}{2}\,dx$$

$$+2\nu\int_\Omega h\left(1 + (1-2k)\,|u|^k\right)|D(u)|^2 + \delta(\beta)\,|A(u)|^2\,dx$$

$$+g\nu\int_\Omega \left|\nabla\left(\sqrt{3\phi(\beta)\beta}b + \sqrt{2/3\delta(\beta)}h\right)\right|^2 dx \leqslant 0 \tag{26}$$

where $\psi = u + 2\nu\nabla \ln h$.

Proof of Lemma 4.1

We multiply the momentum equation by $u + \gamma u |u|^k$ and using the mass equation for h and b and integrate by parts to obtain:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{h|u|^2}{2} + \frac{\gamma}{k+2} h|u|^{k+2}\, dx \\
&+2\nu \int_\Omega h|D(u)|^2 - \gamma \mathrm{div}(hD(u)) \cdot u|u|^k\, dx \\
&+g\int_\Omega \partial_t h^2/6 + b\partial_t h + h\gamma/(3\alpha)\partial_t b + \gamma/(2\alpha)\partial_t b^2\, dx \\
&+g\nu \int_\Omega \beta\gamma/(3\alpha)\nabla b \cdot \nabla h + \beta\gamma/\alpha |\nabla b|^2\, dx = 0
\end{aligned}
\tag{27}
$$

But, sign of terms in red are unknown, we have to get more additional information to conclude: this is achieved with the mathematical entropy, BD-entropy.

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The BD-entropy

Lemma

Let (h, u, b) be a regular solution of (23). Then the following equality holds:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h\,|\psi|^2\,dx + \int_\Omega 2\nu\,|A(u)|^2\,dx$$
$$+\int_\Omega g/6\partial_t h^2 + 2g\nu/3\,|\nabla h|^2 + gb\partial_t h + 2g\nu\nabla b\cdot\nabla h\,dx = 0 \tag{28}$$

Proof of Lemma 4.2

- take the gradient of the mass equation,
- multiply by 2ν and write the terms ∇h as $h\nabla \ln h$ to obtain:

$$\partial_t (2\nu h\nabla \ln h) + \operatorname{div}(2\nu h\nabla \ln h \otimes u) + \operatorname{div}\left(2\nu h\nabla^t u\right) = 0 \qquad (29)$$

- sum Equation (29) with the momentum equation of System (23) to get the equation:

$$\partial_t (h\psi) + \operatorname{div}(\psi \otimes hu) + h\nabla(h/3 + b) + 2\nu \operatorname{div}(hA(u)) \qquad (30)$$

  where $\psi = u + 2\nu\nabla \ln h$ the BD multiplier and $2A(u) = \nabla u - \nabla^t u$ the vorticity tensor.
- multiply the previous equation by ψ and integrate by parts ■

End of the proof of Theorem

Add result of the first lemma to the result of the second lemma multiplied by δ provides finish the proof.

$\blacksquare$

The Grass model

If we assume that the morphodynamic bed-load transport equation is given by:

$$\nabla_x b - v = \mathrm{div}(hu)$$

which means that the sediment layer level evolves as the fluid height. Thus, the model reduces to :

$$\left\{ \begin{array}{lcl} \partial_t h + \mathrm{div}(hu) & = & 0, \\ \partial_t(hu) + \mathrm{div}(hu \otimes u) + gh\nabla\left(\dfrac{h}{3} + b\right) & = & 2\nu \mathrm{div}(hD(u)) \\ & & - \left(\begin{array}{c} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u) \end{array} \right), \\ \partial_t b + \mathrm{div}(hu) & = & 0. \end{array} \right. \quad (31)$$

Mass equation for h and solid transport equation for b gives:

$$b(t,x) = h(t,x) - b_0(x) \quad (32)$$

for some given data b_0.

The Grass model

Existence result under the regularity assumption on $b_0 > 0$ [BGL05]
In spite of the pressure term $h^2/3$, result [BGL05] remains true if we add a friction term $r_0 u + r_1 u|u|$ (that we do not write for simplicity in the below inequalities but required for stability).

[BGL05] *D. Bresch and M. Gisclon and C.K. Lin*, An example of low Mach number effects for compressible flows with nonconstant density (height) limit, *M2AN, 39(3):477–486*, 2005.

The Grass model

The energy equality is:

Lemma

Let (h, u, b) *be a regular solution of (31), then the inequality holds:*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega h\,|u|^2\;dx + g\frac{h^2}{6} + g\frac{b_0^2}{2}\,dx \\ &+2\nu\int_\Omega h\,|D(u)|^2\;dx \leqslant \int_\Omega g\frac{b_0^2}{2}\,dx \end{aligned} \tag{31}$$

The Grass model

The BD-entropy is given by:

Lemma

Let (h, u, b) *be a regular solution of (31), then the inequality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h\,|\psi|^2 + g\frac{h^2}{6}\,dx$$

$$+\int_\Omega 2\nu\,|A(u)|^2\,dx + 2g\nu\int_\Omega \frac{5}{3}\,|\nabla h|^2 \leqslant \int_\Omega g\frac{b_0^2}{2} + g\nu\,|\nabla b_0|^2\,dx \tag{31}$$

Then it is sufficient to have $b_0 \in L^2(0, T; L^2(\Omega)$ to apply obtain the existence result in [BGL05].

Outline

- find appropriate kinematic boundary condition
- Write a 2D numerical code to compare to existing result
- Similar model is written in closed pipes (but no up to date no stability result)

Remark

All this work remains true is we consider INSEs instead of CNSEs

Thank you for your attention